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Redakcioni odbor još jednom koristi priliku da se zahvali na doprinosu svim autorima koji su omogućili održavanje godišnje konferencije «Savremeni problemi u građevinarstvu» i izdavanju Zbornika radova Građevinskog fakulteta u Subotici.

Srdačan pozdrav,

Uredništvo
Subotica, maj 2006.

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UVODNA IZLAGANJA I POZVANI REFERATI:
KEYNOTE LECTURES AND INVITED LECTURES:

Nikola Hajdin

STRUCTURAL MECHANICS AND STRUCTURES
Some author's contributions

1. Introduction

In my very long scientific and professional activity lasting almost 60 years I worked in several branches of structural mechanics. Some of my first contributions in the beginning of the second half of the twentieth century are not actual now but a big part of the contributions even from my first period is according my opinion still actual.

It is my intention to give a review of some of them in the chronological order selecting only such theoretical contributions which are directly connected with my engineering creations representing some development in the technology of structures.

2. A numerical method based on integral equations

After the second world war the analytical methods were dominant in structural mechanics, unfortunately not being able to solve often very complicated problems concerning the different structural forms and geometrical and material non linearity.

It is obvious that some numerical procedures are needed. I tried to find some way how to solve actual problems and find some numerical method which would be possible for practical application using classical calculators.

This was in the period when the computers in engineering practice were unknown thing.

The result of research was a numerical method, first published in 1956 [1], [2], [3], based on integral equations able to solve the problems with use of modest number of linear equations often smaller than the number of equations in the very known method of finite differences.

I would like to explain it on the very simple problem of the elastic torsion:

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=C$$

where

Φ stress function

C given constant.

Adopting a mesh of orthotropic lines (Fig. 2.1), the consisting parts of differential equation along the lines of the mesh

$$\left(\frac{\partial^2\Phi}{\partial x^2}\right)_{y=y_n}=p(x,y_n) \qquad\qquad \left(\frac{\partial^2\Phi}{\partial y^2}\right)_{x=x_m}=q(y,x_m)$$

will be transformed into integral equations.

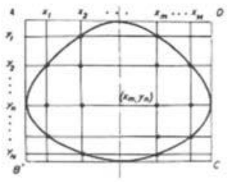

Figure 2.1

Using the numerical integration one gets in the matrix formulation:

$$\Phi = Ap$$
$$\Phi = Bq$$

$$p + q = C$$

And finally the solution of the problem

$$\left(I + B^{-1}\right)p = C$$

The method was applied for solving several problems of different structures and other problems of mechanics like dynamics of vehicles, hydraulics and analysis of arch dams [4], [5]. Majority of big arch dams in former Yugoslavia have been analyzed using this method.

The arch dam Glaznja (Fig. 2.2), one of the biggest in former Yugoslavia, designed in 1968 was analyzed using this method.

Figure 2.2

The method was quoted and used by several authors in Yugoslavia and abroad.

3. Creep of concrete and composite structures

After the second world war a new technology was introduced in the construction of bridges, known as composite bridges, consisting of two materials with different characteristics - concrete and steel - acting together in the structural system.

The concrete differs from steel as material having different relation between the stress and strain. The concrete shows a time dependent strain (deformation) which in the composite action influences stresses in the steel part of structure.

This makes more complicated analysis of the structures. The usual combination of the concrete and steel was: concrete plate on the top of girder combined with the steel underneath the concrete plate (Fig. 3.1).

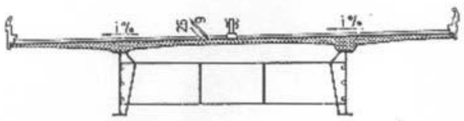

Figure 3.1

It was my ambition to propose something more general: structure with the arbitrary position of the concrete in the cross section.

The paper [6], was published at the beginning of sixties of the last century, see also [7], and practically at the same time I designed a big bridge (with span of 135 m) across the Sava River at Orasje (Croatia) completed in 1968 (Fig. 3.2), following my previously done theoretical investigation [8].

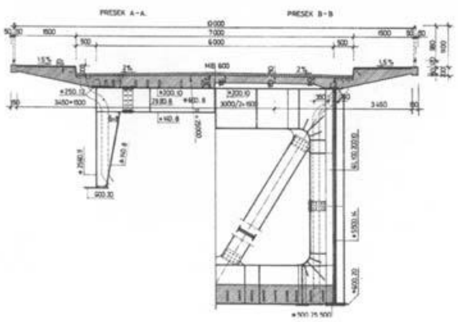

Figure 3.2

This was the first bridge in the word with double composite action i.e. with the top concrete plate and the bottom concrete plate in the zones of supports.

First bridge of this art was constructed in Germany 25 years later.

4. About the thin walled structures

A big part of my scientific activity was oriented to the theory of so called thin walled structures which are extremely important in the structural practice.

They are often basis in the construction of different objects mostly in metal materials. In this topic I wrote more than 30 papers, publications and two books in Springer edition [9], [10], [11].

These papers and books have been cited and used in several hundreds other papers and the books, and used in many universities as the literature for post graduate students.

5. Fatigue of cables in cable stayed bridges

The cable material, a high strength steel, is very sensitive on fatigue. Due to this fact, the amplitude between the maximum and minimum stress is limited. From other side the oscillation of loading due to traffic in the case of railways bridges can be very high.

Because of that fact it was common opinion that the cable stayed bridges are not suitable for railway bridges.

The theoretical studies, I have done, have shown that under certain conditions the application of cable stayed bridges for railway traffic could be possible.

In order to realize this idea some theoretical and experimental studies have been necessary, among others the bending of cables in the anchorage zones.

An analysis of this phenomenon was done [12] taking into account displacements of the anchorage points in the pylon and girder (Fig. 5.1), using new type of cables with the wires in polyethylene tube (Fig. 5.2).

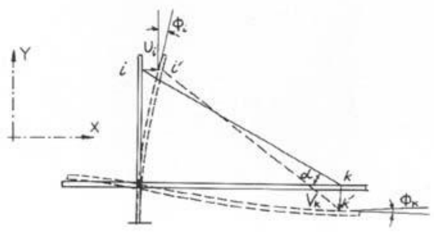

Figure 5.1

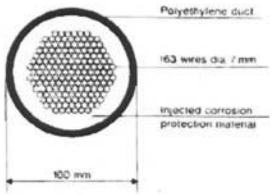

Figure 5.2

The results we obtained have shown that these additional stresses are important and we respected them during the construction of the first cable stayed bridge in the world for the railways traffic only that I designed across the Sava river in Belgrade.

This phenomenon was later considered by others authors during the construction of several bridges of this kind by introducing some additional devices in the anchorage zones.

The railroad bridge across the River Sava between the "Novi Beograd" and "Prokop" stations is 1928 m long in all [13], [14]. It consists of a crossing over the river and approaches on the left and right banks (Fig. 5.3).

Figure 5.3

The approaches on the left bank section are 791.36 m long, the central section above the River Sava and the Winter Harbor is 557.94 m and the approach on the right bank section 578.76 m long.

The main bridge structure – its central section – is a continuous girder (stiffening beam) with spans of 52.74 + 85.00 +254.00 + 50.00 + 64.20 = 555.94 m, with cable stays in the central spans. The stiffening beam consists of two box girders (Fig. 5.4) of a constant height of 4.45 m mutually interlinked by an orthotropic deck, which carries the ballast and tracks.

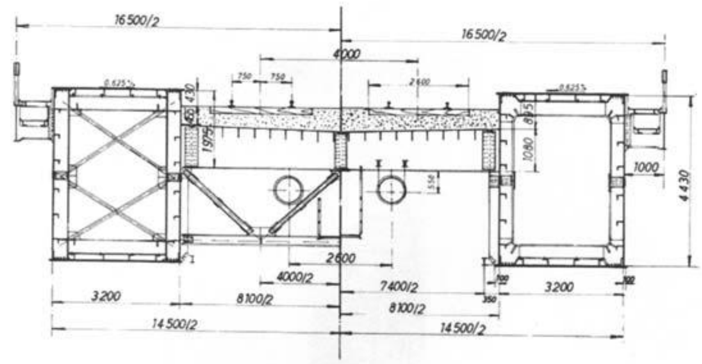

Figure 5.4

On both sides of the main span there is a pair of vertical pylons anchored in the bridge stiffening girder. The cable stays are distributed in two vertical planes supporting the stiffening girder at approximately every fifth of the 254 m span. All the cables are anchored above the supports of 50 m long lateral spans. Adopting BBR system parallel wire cables with high fatigue resistant Hi-Am anchors, along with measures to increase the bridge mass, optimum stress level, excellent cable tension for dead loads and an insignificant influence of cable sag on vertical displacement of the structure, was achieved.

The bridge was completed in 1979.

We should mention that this is the first time that this type of cables were used in Europe. Since that time, up to date, this type of cable has been the dominant form used for cable-stayed bridges in the world.

6. Stress and strain distribution at local points of cable stayed bridges

The elements which differ the cable stayed bridges from other structures are anchorage zones where very high concentration of stresses occur.

The analysis of these stress concentrations which have usually two dimensional character is based on finite element procedure [15]. A detailed analysis have been done on the basis of elastic and elastoplastic model considering the safety margin of element in the case of fully plastic behavior (Fig. 6.1, Fig. 6.2).

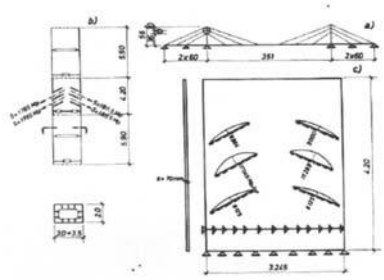

Figure 6.1

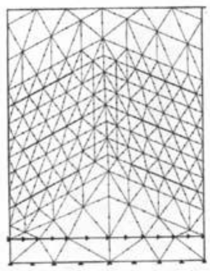

Figure 6.2

This investigation have been used during the construction of the cable stayed bridge across the Danube river in Novi Sad, completed in 1981 [16], [17].

There are other theoretical investigations which have been used in the construction of this bridge.

The main structure of the roadway bridge across the Danube in Novi Sad (Fig. 6.3) is a girder with cable stays. With its 351 m span, it set, at the time of building, the world record for bridges of this type, with pylons and stays in the central plane of the bridge.

Proceeding from the Novi Sad side, (the left bank), the bridge comprises:
a) the approach bank structure which is 301 m long, made of prestressed concrete,
b) the access composite structure of the left bank with spans of 4 x 60 = 240 m,
c) the main steel structure of the girder system with cable stays and spans of 2 x 60+ 351 + 2 x 60 = 591 m,
d) the access composite structure of the right bank with spans of: 3x60 = 180 m. The total length of the bridge is 1312 meters. The bridge is designed to accommodate six traffic lanes.

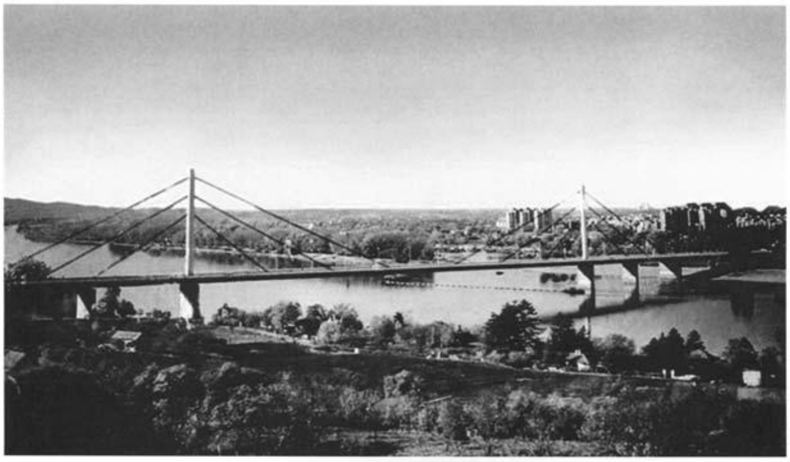

Figure 6.3

The main bridge structure is undoubtedly the most important and most complex part of the entire bridge. The stiffening girder or the main girder of the bridge has a box cross section, trapezoidal in shape (Fig. 6.4).

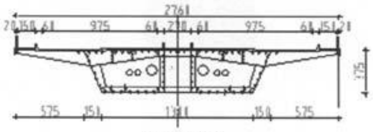

Figure 6.4

The height of the box is 3.8 m, the width of the lower plate is 13.0 m, and of the upper plate 27.48 m, of which the width of 16.0 m is an integral part of the closed cross section. The pylons of the bridge are above the piers, at the ends of the main span, positioned in the axis of the bridge and fixed in the stiffening girder. Three groups with 4 parallel wire cables are arranged in a single plane having a harp configuration. They are spaced along the main span of the bridge at distances of 54+48+48 m symmetrically on both sides.

This bridge was destructed during the NATO bombing (Fig. 6.5) and reconstructed in fully original shape at the end of the year 2005 [18], [19], [20].

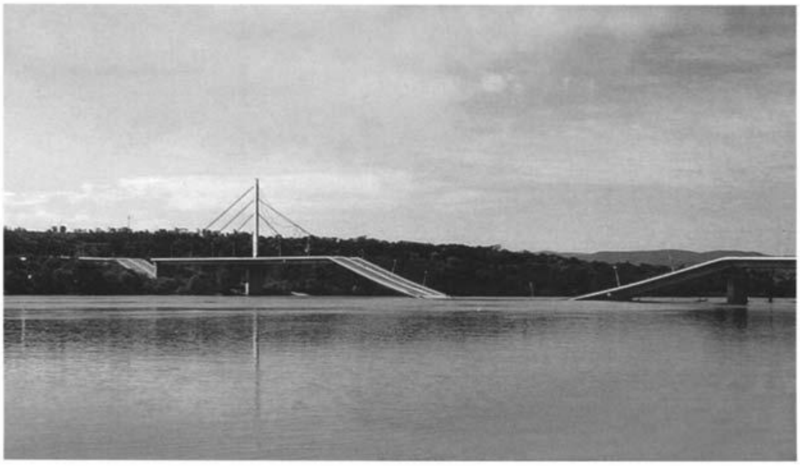

Figure 6.5

7. Patch loading - theoretical and experimental investigations

The stability problems and ultimate load behavior of steel plate girders have attracted a lot of attention during the last few decades. The behavior of the plate girder subjected to patch load or partially distributed load on the flange in the plane of a web without vertical stiffener bellow the load was also intensively investigated.

Our research was concentrated on the behavior of girders with longitudinal stiffeners made on series of tests on plate girders (Fig. 7.1)

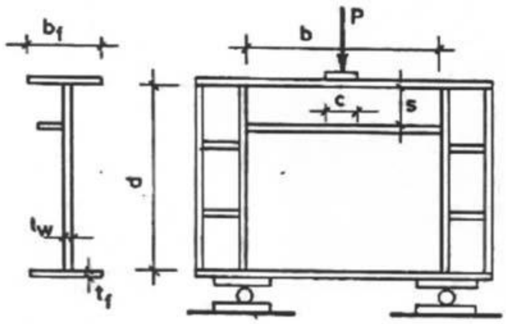

Figure 7.1

Theoretically [21] some model was proposed leading to the value of ultimate load.

We were included in the common research with the scientists in England and Czech Republic [22] .

As result of our investigations was a criterion for the ultimate load which was used in the British standard.

8. Ship impact on structures (bridges)

Ship impact has attracted a considerable attention of engineers, mainly due to safety reasons in bridge design.

In the last few years I have studied with my collaborators various problem concerning ship impact problem on rivers and canals [23], [24], [25], [26]. The main goal of our research has been an estimate of impact actions on civil engineering structures which can be used as reliable base for the analysis of impacted structure.

At the beginning the effort has been concentrated on bow impact problem and later on sideway impact.

A considerable crushing of ship bow structure during a collision takes place. In the crushed zone large rotations, displacements and even large strain components of individual structural elements are present. However in the most cases the crushed zone is relatively small in comparison to the length of a vessel.

Crushing characteristics of a ship bow structure have been analyzed on the basis of the Maier-Doernberg experimental research. The reaction forces due to the collapse mechanism have been divided on deck and bottom structures.

The deck or bottom structure is modeled as an assemblage of finite number of folded sections (Fig. 8.1).

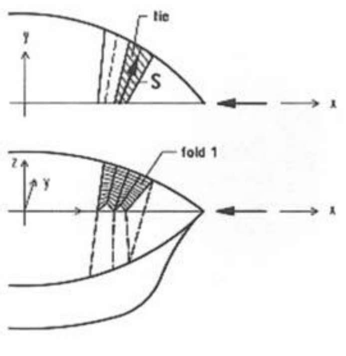

Figure 8.1 Formation of folds in the deck's plate

Each folded section is divided in two transverse elements and one longitudinal element. The later is assumed to buckle elastoplastically out of deck's plane.

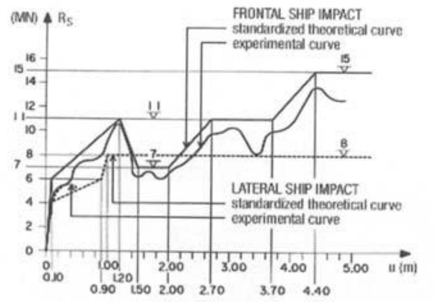

Figure 8.2 deformation behaviour of ship's hull

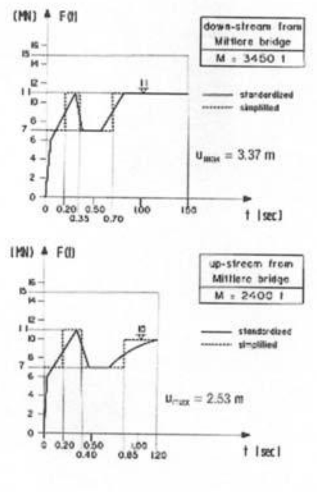

Figure 8.3 Relevant impact function F(t) for frontal ship impact

Trough supposed art of deformation a corresponding total reaction force has been obtained for each step of deformation that means displacement of the bow into longitudinal direction represented as the F-d function and shown on the Figure 8.2.

The last step is the solution of the dynamic equation on the basis of the relation between the force and deformation (Fig. 8.3).

This analysis has been used in the calculation of several bridge's piers in the rivers in Switzerland.

9. Last design achievement: bridge across the River Vistula

Last design achievement was the roadway bridge across the River Vistula in Plock in Poland, 1st prize at an anonymous international competition (Fig. 9.1), cable stayed bridge with the span 370 m and cables in one single vertical plane [27], [28].

The total length of the bridge is 1200 meters, of which 615 meters is the length of the main part of the bridge over the Vistula riverbed and 585 meters the length of the access part of the bridge above the inundation basin. The main bridge structure is a symmetrical steel structure, a cable-stayed bridge, composed of: a continuous girder (with 2 x 60 + 375 +2 x 60 meter spans), cable stays and two pylons.

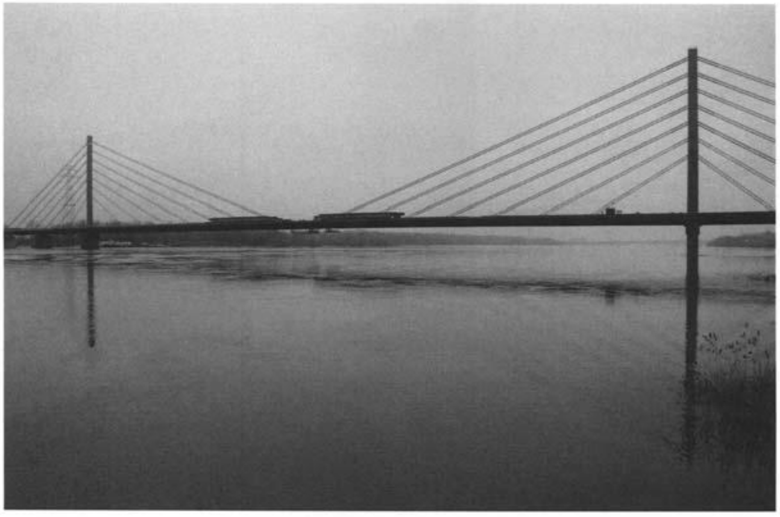

Figure 9.1

The bridge girder has a torsionally stiff three-cell cross section of trapezoidal shape, (height 3.5 m, lower plate width 13.0 m, upper plate width 16.5 m), cantilever arms 5.5 m wide. The pylons to which the cable stays transmit their tensile force are made of steel and fixed in the girder of the bridge. The cable stays are placed in the central, vertical plane of the bridge, in what is referred to as modified harp distribution. Each cable stay consists of two individual cables (ropes), at axial distance of 750 mm.

The bridge was completed in the year 2005.

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MODELIRANJE GRANICNIH USLOVA U MEHANICI FLUIDA POMOCU FRAKCIONIH IZVODA

Vladan D. Djordjevic[1]
UDK:532.517

Fenomeni strujanja koji se tretiraju u oblasti mehanike fluida su veoma raznovrsni, i zastupljeni su kako u tehnici, tako i u prirodi koja nas okruzuje. Oni se odlikuju sirokim spektrom vremenskih i prostornih razmera. Strujanja mogu biti viskozna i neviskozna, laminarna i turbulentna, izotermska i neizotermska, mogu se odnositi na nestisljiv i stisljiv fluid, fluid koji je elektricki provodljiv, ili neprovodljiv, itd. S tim u vezi postoji i citav niz veoma razlicitih pocetnih i granicnih uslova koji se koriste prilikom resavanja osnovnih jednacina kojima se opisuje strujanje fluida.

U radu ce se prikazati pokusaj da se razliciti granicni uslovi koje ima smisla koristiti pri resavanju jednog te istog problema strujanja objedine u jedinstveni granicni uslov koji bi bio definisan pomocu frakcionog izvoda. Teziste rada ce biti na modeliranju granicnog uslova klizanja razredjenog gasa prilikom strujanja u mikrokanalima. Pokazace se kako se to moze postici jednom malom modifikacijom frakcionog izvoda tipa Caputo, kod kojeg red izvoda nije konstantan, nego na odredjeni nacin zavisi od lokalne vrednosti Knudsenovog broja u kanalu. Ovaj broj predstavlja meru razredjenosti gasa, jednak je nuli kod klasicnog strujanja gasa koje se tretira u mehanici kontinuuma, a moze imati veoma velike vrednosti kod tzv. slobodnog molekularnog kretanja. U kanalima mikro-, ili nano-razmera, koji se danas koriste u tzv. MEMS tehnologijama, svi rezimi strujanja koji se odnose na vrednost Knudsenovog broja mogu doci do izrazaja kod jednog te istog kanala, pa zato rezultati koji ce biti predstavljeni, a koji se odnose na primenu frakcionog izvoda u modeliranju granicnog uslova klizanja gasa, mogu imati veoma korisne primene.

---

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THE MESHLESS ANALOG EQUATION METHOD. A NEW HIGHLY ACCURATE MESH-FREE METHOD FOR SOLVING LINEAR AND NONLINEAR PDEs

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Summary: *A new purely meshless method to solve linear and non linear PDEs encountered in mathematical physics and engineering is presented. The method is based on the concept of the analog equation of Katsikadelis, hence its name meshless analog equation method (MAEM), which converts the original equation into a simple solvable substitute one of the same order under a fictitious source. The fictitious source is represented by Multiquadrics Radial Basis Functions (MQ-RBFs). Integration of the analog equation allows the approximation of the sought solution by new RBFs. Then inserting the solution into the PDE and BCs and collocating at the mesh-free nodal points yields a system of equations linear or non linear depending on the differential equation, which permit the evaluation of the expansion coefficients. The method exhibits key advantages of over other RBF collocation methods as it is highly accurate and gives well conditioned coefficient matrix, which is always invertible. The accuracy is increased using optimal values of the shape parameters of the multiquadrics by minimizing the potential that produces the PDE. Without restricting its generality, the method is illustrated by applying it to the general second order elliptic or quasi-elliptic PDE. The studied examples demonstrate the efficiency and high accuracy of the developed method.*

Key words: *meshless method, elliptic partial differential equations, radial basis functions, analog equation method, anisotropic, inhomogeneous, nonlinear*

1. INTRODUCTION

The interest in mesh-free methods to solve PDEs has grown noticeably in the past 20 years. This is mainly due to the fact that the traditional methods FDM, FEM and BEM, although effective, exhibit crucial drawbacks. The FDM is ineffective for domains with complex geometry. The FEM requires mesh generation over complicated 2D and especially 3D domains, which is a very difficult problem and may require long time, in some cases weeks, to create a well behaved mesh. The BEM circumvents the domain discretization for 2D problems as it reduces the dimensions by one, but encounters the domain discretization on the surface in 3D problems. Moreover, the convergence rate of the traditional methods is of second order. The mesh-free MQ-RBFs (multiquadric radial

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basis functions) method developed by Kansa [1, 2] has attracted the interest of the investigators, because it is truly meshless, very simple to implement and enjoys exponential convergence. The primary disadvantage of the MQ scheme is that it is global, hence the coefficient matrices resulting from this scheme are full and suffer from ill-conditioning, particularly as the rank increases. Extended research has been performed by many investigators to circumvent this drawback and several techniques have been proposed to improve the conditioning of the matrix, [3], which, however, complicate the implementation of the MQ-RBFs method and render it rather problem dependent. Moreover, although the performance of the method depends on the shape parameter of MQs, there is no widely accepted recipe for choosing the optimal shape parameters. Therefore, extended research is ongoing to optimize these parameters [4]. Nevertheless, all these quantities are chosen arbitrarily or empirically.

The presented here new meshless RBFs method, the MAEM, overcomes the drawbacks of the standard MQ-RBFs method. The method is based on the concept of the analog equation of Katsikadelis, according to which the original equation, regardless of its being linear or non linear, is converted into a substitute linear equation, *the analog equation*, under a fictitious source. The fictitious source is represented by radial basis functions series of multiquadric type. Integration of the analog equation yields the sought solution as series of new radial basis functions. To make this idea more concrete we consider the following elliptic BVP

$$Nu = g \quad \text{in } \Omega \tag{1}$$

$$Bu = \bar{g} \quad \text{on } \Gamma \tag{2}$$

$u = u(\mathbf{x})$ is the sought solution of eqn (1) and N , B linear or nonlinear operators. If $\mathcal{L}$ is another linear operator of the same order as N , we obtain

$$\mathcal{L}u = b \quad \text{in } \Omega \tag{3}$$

where $b = b(\mathbf{x})$ is an unknown fictitious source. Eqn (3) under the boundary condition (2) can give the solution of the problem, if the fictitious source $b(\mathbf{x})$ is first established. To this end, the fictitious source is approximated by MQ-RBFs series. Thus, we can write

$$\mathcal{L}u = \sum_{j=1}^{M+N} a_j f_j \quad \text{in } \Omega \tag{4}$$

where $f_j = \sqrt{r^2 + c^2}$, $r = \| \mathbf{x} - \mathbf{x}_j \|$ represents the Euclidean distance of point $\mathbf{x}$ from the collocation point $\mathbf{x}_j$ and M, N represent the number of collocation points inside Ω and on Γ, respectively. Eqn (4) is integrated to yield the solution

$$u \simeq \sum_{j=1}^{M+N} a_j \hat{u}_j \tag{5}$$

where $\hat{u}_j = \hat{u}_j(r)$ is the solution of

$$\mathcal{L}\hat{u}_j = f_j \tag{6}$$

Since $\mathcal{L}$ is arbitrary, it is chosen so that the solution of Eqn (6) can be readily established, e.g. if $\mathcal{L}$ is of the second order, we can choose $\mathcal{L} = \nabla^2$. Subsequently, the solution (5) is inserted into the PDE (1) and BC (2) to yield

$$\sum_{j=1}^{M+N} N\,[\hat{u}_j(\mathbf{x})a_j\,] = g \quad \text{in } \Omega \tag{7}$$

$$\sum_{j=1}^{M+N} B\,[\hat{u}_j(\mathbf{x})a_j\,] = \bar{g} \quad \text{on } \Gamma \tag{8}$$

Collocation of Eqns (7) and (8) at the $M + N$ nodal points (see Figure 1), yields a system of linear or nonlinear equations, which permit the evaluation of the expansion coefficients.

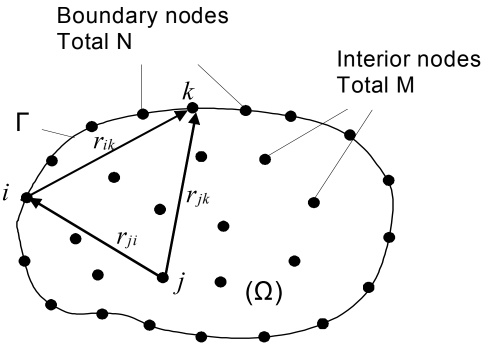

Figure 1

The major advantage of the presented formulation is that it results in well behaved coefficient matrices, which can be always inverted. Moreover, since the accuracy of the solution depends on a shape parameter of the MQs, the position of the collocation points and the integration constants of the analog Eqn (6), a procedure is developed to optimize these parameters by minimization of the functional that produces the PDE as Euler-Lagrange equation [4] under the inequality constraint that the condition number of the coefficient matrix ensures invertibility. This procedure, which minimizes the error of the solution, requires the evaluation of a domain integral during the minimization process, which is facilitated by converting it to a boundary integral using DRM. The method is illustrated by applying it to the solution of the general second order elliptic PDE. Several examples are studied, which demonstrate the efficiency and accuracy of the method.

2. LINEAR ELLIPTIC PDEs

The method is demonstrated first by applied to the partial differential equation

$$A\,u_{,xx} + 2Bu_{,xy} + Cu_{,yy} + Du_{,x} + Eu_{,y} + Fu = g(\mathbf{x}) \quad \text{in } \mathbf{x} \in \Omega \tag{9}$$

subject to the boundary conditions

$$u = a(\mathbf{x}), \quad \mathbf{x} \in \Gamma_u \tag{10a}$$

$$ku + \nabla u \cdot \mathbf{m} = g(\mathbf{x}), \quad \mathbf{x} \in \Gamma_m \tag{10b}$$

where $\Gamma = \Gamma_u \cup \Gamma_m$ is the boundary of Ω, which may be multiply-connected; $u = u(\mathbf{x})$ is the unknown field function; $A, B, \ldots, F$ position dependent coefficients satisfying the ellipticity condition $B^2 - AC < 0$, $\mathbf{m} = (An_x + Bn_y)\mathbf{i} + (Bn_x + Cn_y)\mathbf{j}$ is a vector in the direction of the con-normal on the boundary. Finally, $k(\mathbf{x})$, $a(\mathbf{x})$ and $g(\mathbf{x})$ are functions specified on Γ. We consider the functional [5].

$$J(u) = \int_{\Omega} \left[\frac{1}{2} \left(A u_{,x}^2 + 2B u_{,x} u_{,y} + C u_{,y}^2 - F u^2 \right) + g u \right] d\Omega + \int_{\Gamma} \left(\frac{1}{2} k u^2 - g u \right) ds \tag{11}$$

We can easily show that the condition $\delta J(u) = 0$ yields the boundary value (9), (10) provided that

$$A_{,x} + B_{,y} = D, \qquad B_{,x} + C_{,y} = E \tag{12a,b}$$

Therefore, the solution of Eqn (9) under the boundary conditions (10a,b) make $J(u) = \min$.

The boundary value problem (9), (10) under the conditions (12) for suitable meaning of the coefficients occurs in many physical problems such as thermostatic, elastostatic, electrostatic and seepage problems, where the involved media exhibit heterogeneous anisotropic properties.

2.1 THE MAEM SOLUTION

The analog equation is obtained from eqn (3), if we take $\tilde{L} = \nabla^2$. Thus we have

$$\nabla^2 u = b(\mathbf{x}) \tag{13}$$

and (6) becomes

$$\nabla^2 \hat{u}_j = f_j \tag{14}$$

which for $f_j = \sqrt{r^2 + c^2}$ yields after integration

$$\hat{u}_j = \frac{1}{9} f^3 + \frac{1}{3} f c^2 - \frac{c^3}{3} \ln(c + f) + G \ln r + F \tag{15}$$

where $G = 0$ for $r = 0$, otherwise it is arbitrary. As we will see, the arbitrary constants G, F play an important role in the method, because together with the shape parameter c they control the conditioning of the coefficient matrix and the accuracy of the results. Thus, the solution is approximated by

$$u \simeq \sum_{j=1}^{M+N} a_j \hat{u}_j \tag{16}$$

and it is forced to satisfy the governing equation and the boundary conditions. For this purpose, it is inserted into eqns (9) and (10) to yield the system of linear equations

$$\mathbf{Aa} = \mathbf{b} \tag{17}$$

where

$$\mathbf{A} = \left[L\hat{u}_{ij} \quad B\hat{u}_{ij} \right]^T, \ \mathbf{b} = \left\{ \mathbf{g} \quad a \quad g \right\}^T \tag{18}$$

in which B is the operator defined by eqns (10a,b).

The approximation (16) with the new radial basis functions $\hat{u}_j$ is better than the conventional one with $f_j = \sqrt{r^2 + c^2}$, because they can accurately approximate not only the field function itself but also the first and second derivatives. This shown in Fig. 2.

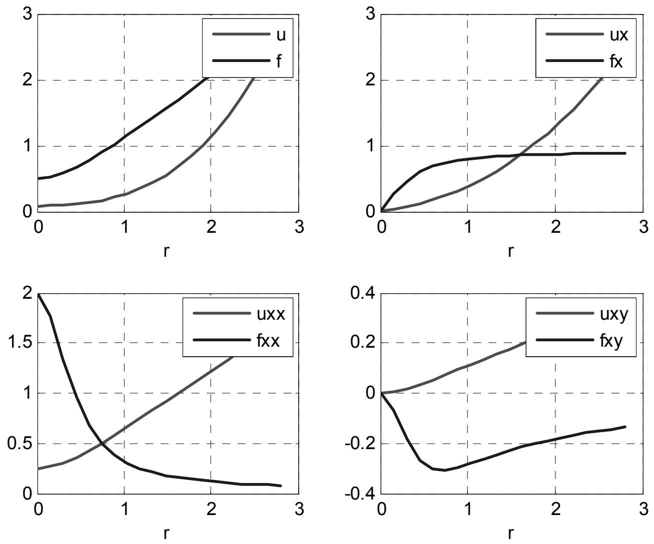

Figure 2: Variation of $f(r)$, $\hat{u}(r)$ and their derivatives along the line $y = 0.5x$
($c = 0.5$, $G = 1.\text{e}-2$, $F = 0$).

3. OPTIMAL VALUES OF THE PARAMETERS

The coefficients a_j evaluated from eqn (17) can be used to obtain optimal values of the shape parameter centers of the multiquadrics and the integration constants G, F by minimizing the functional (11). The evaluation of the domain integral is facilitated, if it is converted to boundary line integral using DRM [5]. Thus, denoting by

$$R(\mathbf{x}) = \frac{1}{2}\left(A u_{,x}^2 + 2B u_{,x} u_{,y} + C u_{,y}^2 - F u^2\right) + g u \tag{19}$$

and approximating the integrand of the domain integral by

$$R(\mathbf{x}) \approx \sum_{j=1}^{M} \bar{a}_j \hat{u}_j(r) \tag{20}$$

we obtain

$$\int_\Omega R(\mathbf{x}) d\Omega \approx \sum_{j=1}^{M} \bar{a}_j \int_\Omega \hat{u}_j(r) d\Omega \tag{21}$$

Application of (20) at the collocation points yields

$$\bar{\mathbf{a}} = \mathbf{U}^{-1}\mathbf{R}, \quad \mathbf{U} = [\hat{u}(r_{ji})], \quad \mathbf{R} = \{R(x_i)\} \tag{22}$$

Subsequently, using the Green' reciprocal identity

$$\int_\Omega \left(v \nabla^2 u - u \nabla^2 v\right) d\Omega = \int_\Gamma \left(v u_{,n} - u v_{,n}\right) ds \tag{23}$$

for $v = 1$ and $u = \hat{w}_j$, where $\hat{w}_j$ is a particular solution $\nabla^2 \hat{w}_j = \hat{u}_j$ we obtain

$$\int_\Omega R(\mathbf{x}) d\Omega \approx \mathbf{1}^T \hat{\mathbf{Q}} \; \mathbf{U}^{-1}\mathbf{R} \tag{24}$$

where $\hat{Q}_{jk} = \int_{\Gamma_k} \hat{w}_{j,n}(r_{jk})ds$ and $\mathbf{1}^T = \{1 \quad 1 \quad \cdots \quad 1\}$.

It is apparent that the functional $J(u)$ depends on the following sets of parameters:

(i). The shape parameter c and the arbitrary constants G, F.

(ii). The $2M + 2N$ coordinates x_j, y_j of the centers.

Therefore, we can search for the minimum using various levels of optimization depending on the design parameters that we wish to be involved in the optimization procedure. Although, the functional $J(u)$ is quadratic with respect to a_j, the inclusion however of c and x_j, y_j requires direct minimization methods for nonlinear objective functions.

4. NONLINEAR EQUATIONS

For nonlinear PDEs the procedure is exactly the same as for linear equations. The only difference is that equation corresponding to eqn (17) for the evaluation of the coefficients is a non linear algebraic equation.

5. NUMERICAL EXAMPLES

Example 1. As a first example we obtain the solution of the following boundary value problem for complete second order linear partial differential equation

$$(1 + y^2)u_{xx} + 2xyu_{xy} + (1 + 2x^2)u_{yy} + xu_x + yu_y + u = 7x^2 - 5xy + 5y^2 + 4 \text{ in } \Omega$$

where Ω is the ellipse with semi-axes $a = 5, b = 3$.

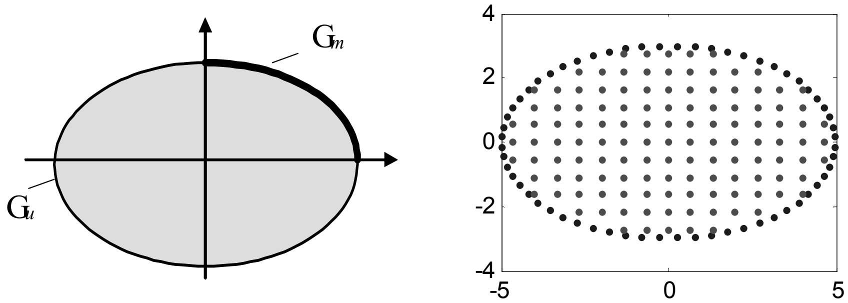

Figure 3: Elliptic domain and nodal points

Three types of boundary conditions have studied

(i) $u = \alpha(\mathbf{x})$ on Γ (Dirichlet)

(ii) $\nabla u \cdot \mathbf{n} = g(\mathbf{x})$ on Γ (Neumann)

(iii) $\nabla u \cdot \mathbf{n} = g(\mathbf{x})$ on Γ_m, $u = \alpha(\mathbf{x})$ on Γ_u (mixed)

where $\Gamma_m = \{y = b\sqrt{1 - x^2/a^2},\ 0 \le x \le a\}$, $\Gamma_u = \Gamma - \Gamma_m$ and

$$a(\mathbf{x}) = x^2 - xy + y^2$$

$$g(\mathbf{x}) = \left[\frac{(1+y^2)x}{a} + \frac{xy^2}{b} \right](2x - y) + \left[\frac{x^2 y}{a} + \frac{(1+2x^2)y}{b} \right](-x + 2y)$$

The analytical solution is $u_{exact} = x^2 - xy + y^2$. The results obtained with $N = 60$, $M = 125$, $c = 7$, $G = 5\mathrm{e}-9$, $F = 0$ are shown in Fig. 4. Fig. 5 shows the convergence of $RMS = \frac{1}{m}\sqrt{\sum_{i=1}^{m} \{[u(i) - u_{exact}(i)]/u_{exact}(i)\}^2}$ with increasing shape parameter. In all three cases the computed results are practically identical with the exact ones.

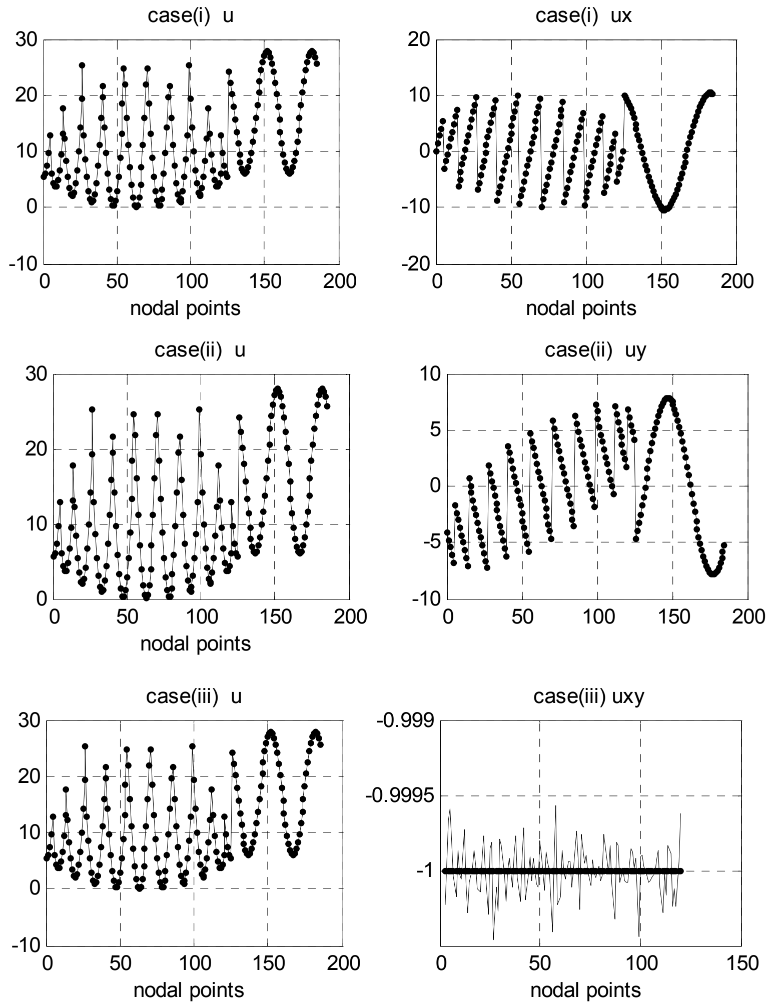

Figure 4: Nodal values of the solution and its derivative in Example 1. Solid line: computed.

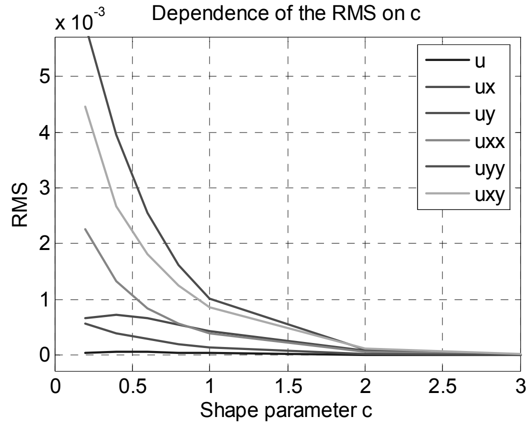

Figure 5: Dependence of RMS on c in Example 1 (case i)

Example 2. As an example of non linear equation we study the following BVP describing the steady state heat conduction problem in a plane body with non linear material properties

$$k\nabla^2 u + \frac{k_0 b}{u_0}\left(u_x^2 + u_y^2\right) = f(x,y) \quad \text{in } \Omega = (0,1)\times(0,1)$$

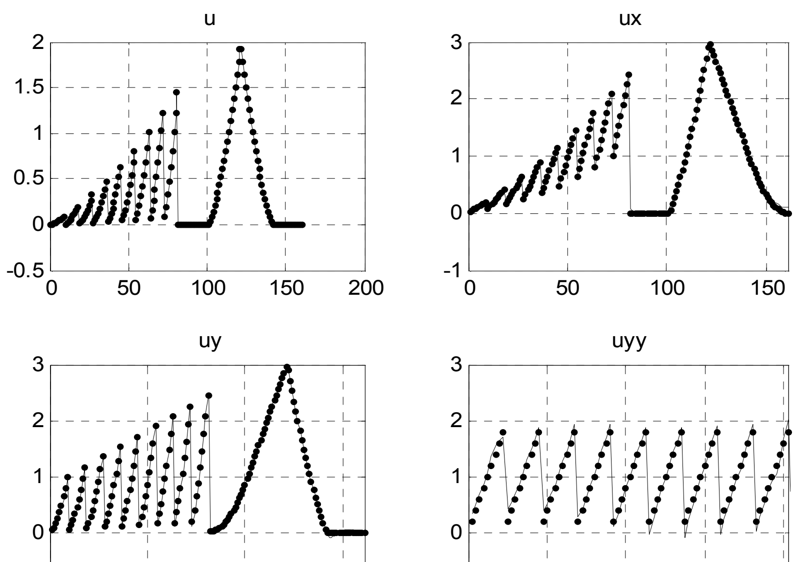

Figure 6: Nodal values of the solution and its derivatives in Example 2. Solid line: computed.

under mixed boundary conditions

$u_n(x,0) = -2x^2, \quad 0 \le x \le 1$ $\qquad$ $u_n(1,y) = 2(1+y), \quad 0 \le y \le 1$

$u_n(x,1) = 2x(1+x), \quad 0 \le x \le 1$ $\qquad$ $u_n(0,y) = 0, \quad 0 \le y \le 1$

The employed data are:

$$f = 2k(x+y) + \frac{k_0 b}{u_0}(x^4 + y^4 + 4xy^3 + 4x^3y + 8x^2y^2), \quad k_0 = 1, \ u_0 = 300, \quad b = 3$$

The results obtained with $c = 0.888$, $G = 1e-9$, $F = 0$ are shown in Fig. 6.

6. CONCLUSIONS

A new truly meshless method, the MAEM (Meshless Analog Equation Method) is developed for solving PDEs, which describe the response of physical systems. The method is based on the concept of the analog equation, which converts the original equation into a Poisson's equation. Using MQ-RBFs to approximate the fictitious source and integrating the analog equation lead to the approximation of the sought solution by new RBFs, which have key advantages over the direct MQ-RBFs. Namely

- The condition number of the coefficient matrix is controlled, thus it can be always inverted to give the RBFs expansion coefficients.
- The method gives accurate results, because the new RBFs approximate accurately not only the solution itself but also its derivatives.
- Optimum values of the shape parameters, centers of RBFs and integration constants can be established by minimizing the potential that yields the PDE. Therefore, the uncertainty of choice of shape parameter is circumvented. It was also observed from the studied examples that a regular mesh of nodal point gives good results and the solution was not sensitive to the position of the RBFs centers.
- The method depends only on the order of the differential operator and not on the specific problem.

Moreover, as other RBFs methods:

- It is truly meshless, hence no domain (FEM) or boundary (BEM) discretization and integration is required. It also avoids establishment of fundamental solutions and evaluation of singular integrals.
- The method can be in a straightforward manner employed for the solution of problems in higher dimensions or other type PDEs (parabolic and hyperbolic).

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BEZMREŽNI ANALOGNI METOD JEDNAČINA. NOVI, VEOMA TAČAN, BEZMREŽAN METOD ZA REŠAVANJE LINERANIH I NELINERANIH PARCIJALNIH DIFERENCIJALNIH JEDNAČINA

Rezime: *U radu je prikazan novi, potpuno, bezmrežni metod za rešavanje lineranih i nelineranih parcijalnih diferncijalnih jednačina koje se susreću u matematičkoj fizici i inženjerstvu. Ovaj metod je zasnovan na konceptu Kastikadelisovih analognih jednačina, po kome je i dobio ime: “Bezmrežni analogni metod jednačina (MAEM)”, koji pretvara početnu jednačinu u jednostavniju, rešivu, zamenjujuću jednačinu istog reda sa fiktivnom osnovom. Fiktivna osnova je predstavljena preko multikvadratnih radijalnih baznih funkcija (Multiquadrics Radial Basis Functions ili MQ-RBFs). Integrisanje analogne jednačine dozvoljava aproksimaciju traženog rešenja preko novih RBF-ja. Uvrštavanjem rešenja u PDJ i granične uslove i njihovim grupisanjem u nediskretizovanu tačku dobija se linearni ili nelinerani sistem jednačina, zavisno od diferencijalne jednačine, koji omogućava izračunavanje koeficijenata razvijanja. Metod pokazuje ključne prednosti nad drugim metodama grupisanja preko RBF-ja kroz svoju tačnost i matricu koeficijenata koja je uvek dobro definisana i regularna. Tačnost se povećava sa izborom optimalnih vrednosti parametara oblika multikvadratura kroz minimizaciju potencijala koji se generiše u PDJ. Bez umanjenja njegove generalizacije, metod je prikazan kroz primenu na opštoj eliptičnoj ili kvazi-eliptičnoj PDJ drugog reda. Prikazani primeri demonstriraju efikasnost i veliku tačnost razvijenog metoda.*

Ključne reči: *bezmrežni metod, eliptične parcijalne diferencijalne jednačine, radijalne bazne funkcije, analogni metod jednačina, anizotropija, nehomogenost, nelinearnost*

DYNAMICS OF SANDWICH STRUCTURES

(Invited Plenary Lecture)

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Summary: *A survey of models and dynamics of sandwich structures composed of a number of plates, beams or belts with different properties of materials and discrete layer properties are presented and mathematically described. The constitutive stress-strain relations for materials of the sandwich structure elements are presented for different properties: elastic, viscoelastic and creeping. The characteristic modes of the sandwich structure vibrations are obtained and analyzed for different kind of materials and structure composition. The visualization of the characteristic modes and amplitude forms are presented. Structural analysis of sandwich structure vibrations are done.*

Key words: *Dynamics, sandwich structure, plate, beam, visco elasticity, relaxation kernel, structural analysis, fractional order derivative, moving sandwich double belt system, two- and multi-frequency regimes, vibrations.*

1. INTRODUCTION

Plates, beams and belts have been extensively used as structural elements in many industrial applications. Investigation of vibrations of plates dates back to the 19th century. There had been a great amount of research and literature presented over the last century. The problem of free vibrations of a circular plate was first investigated by Poisson (1829) [25]. Rayleigh (see Ref. reprint 1945) [27] presented a well known general method of solution to determine the resonant frequencies of vibrating structures. The method was improved by Ritz [26] assuming a set of admissible trial functions. This approach is one of the most popular approximate methods for vibration analysis of the plates, shels, beams. There have been extensive studies of vibrations of plates for various shapes, boundary and loading conditions for nearly two centuries. Interested readers are refereed to excellent reviews of Leissa (1987) [23] and Liew et all. (1995) [24] of this class of problems and to the list of references.

Mechanics of hereditary medium (material) is presented in scientific literature in fundamental monographs by Rabotnov, Yu.N. [28], Rzhanitsin, A.R.[29], Savin G. N. [30], Ruschisky Yu. Ya and O.A.Gorosko (see Gorosko and Hedrih (2001)[2]) and it is widely used in engineering analyses of strength and deformability of constructions made of new construction materials with hereditary properties. This field of mechanics is being intensively developed. Nowadays scale of utilization of these materials can be compared with that of using metals. The book by Enelund (1996) [1] contains some applications with elements of fractional calculus in Structural Dynamics.

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The fast development of science of material and experimental mechanics, of methods of numerical analysis, led to the creation of different models of real material bodies and methods for studying dynamics and processes which happen in them during the transduction of disturbance through deformable bodies. In the process of creating a real body model certain simplifications and approximations are done [28,29,30]. There also exist different approaches to creating real body models. One such approach is represented by a model of discrete system of material points which are connected by certain ties, and the number of which is then increased to create a continuum [5,18], the motion and deformable wave propagation of which was then described by using partial differential equations [6,7,8,9]. And then, due to the impossibility of solving them analyticaly, the approximation method was used for the purpose. Methods of discretization of systems of partial differential equations and methods of physical discretization of continuum were used. Computers were used for obtaining numerical solutions.
In an attempt to make a selection of authors who gave significant copntributions to the knowledge on deformable body dynamics we came to a conclusion that it would require an entire review paper, which is not the goal of this paper so we shall restrict ourselves to citing authors on whose papers we directly rely.

2. DISCRETE CONTINUUM MODELS AND ELEMENTS

In this paper we shall use three basic models of sandwich structures with light constraint elements in the form of layer between plates, or beams or belts. We shall define sandwich structures as a multiplate system, or multibeam system or multi belt system in which deformable bodies (plates, or beams or belts) are interconnected by light standard constraint elements [2] which have the ability to resist axial deformation under static and dynamic conditions.
Basic elements of sandwich structures in multi deformable body systems are:
A* Material deformable bodies in the form of: A.a* thin plates [26]; A.b* thin beams [26], and A.c* elastic belts [11], and every with infinite number degree of freedom.
B* Light standard constraint element [2] of negligible mass in the form of axially stressed rod without bending, and which has the ability to resist deformation under static and dynamic conditions; Constitutive relation between restitution force $\mathbf{P}$ and elongation w or v can be written down in the form $f_{psr}\left(\mathbf{P},\dot{\mathbf{P}},w,\dot{w},\mathsf{D},\mathsf{J},n,c,\tilde{c},\mu,c_\alpha,T,U,......\right)=0$, where D and J are differential integral operators (see Refs. [2, 3, 14,15]) which find their justification in experimental verifications of material behavior [1, 28, 29, 30], while $n,c,\tilde{c},\mu,c_\alpha$ .... are material constants, which are also determined experimentally.
For every single light standard constraint element of negligible mass, we shall define dynamic constitutive relation as deterministic change of forces with distances and changes of distances in time, with accuracy up to constants which depends on the accuracy of their determination through experiment.
The accuracy of those constants laws and with them the equations of forces and elongations will depend not only on knowing the nature of object, but also on our having the knowledge necessary for dealing with very complex stress-strain relations. In this

paper we shall use three such light standard constraint elements, and they will be (for more see Refs. [2] and [18]):

B.1* ***Light standard ideally elastic constraint element*** for which the stress-strain relation for the restitution force, as the function of element axial elongation, is given by a linear relation of the form

$$\mathbf{P} = -cy\,, \tag{1}$$

where c is a rigidity coefficient or an elasticity coefficient. In natural state, non-stressed state, force and deformation of such elemnt are equal to zero.

B.2* ***Light standard hereditary constraint element*** (see Ref. [2]) for which the stress-strain relation for the restitution force as the function of element elongation is given by a relation:

B.2. a* in *differential form*:

$$\mathsf{D}\mathbf{P} = \mathsf{C}y \qquad \text{or} \qquad n\dot{P}(t) + P(t) = ncy(t) + \tilde{c}y(t) \tag{2}$$

where, the following differential operators are introduced:

$$\mathsf{D} = n\frac{d}{dt} + 1 \quad \text{and} \quad \mathsf{C} = nc\frac{d}{dt} + \tilde{c}\,. \tag{3}$$

and n is a relaxation time and $c, \tilde{c}$ are rigidity coefficints – momentary and prolonged one.

B.2. b* in *integral form*

$$P(t) = c\left[y(t) - \int_0^t \mathsf{R}(t-\tau)y(\tau)d\tau\right], \tag{4}$$

where $\mathsf{R}(t-\tau) = \dfrac{c-\tilde{c}}{nc}e^{-\frac{1}{n}(t-\tau)}$ is *relaxation kernel* (or resolvente). (5)

B.2. c* in *integral form*

$$y(t) = \frac{1}{c}\left[P(t) + \int_0^t \mathsf{K}(t-\tau)P(\tau)d\tau\right], \tag{6}$$

where $\mathsf{K}(t-\tau) = \dfrac{c-\tilde{c}}{nc}e^{-\frac{\tilde{c}}{nc}(t-\tau)}$ is *kernel of rheology* (or retardation). (7)

B. 3* ***Light standard creep constraint element*** [5, 18] for which the stress-strain relation for the restitution force as the function of element elongation is given by fractional order derivatives in the form

$$P(t) = -\left\{c_0 w(t) + c_\alpha \mathsf{D}_t^\alpha[w(t)]\right\} \tag{8}$$

where $\mathsf{D}_t^\alpha[\bullet]$ is operator of the α^{th} derivative with respect to time t in the following form:

$$\mathsf{D}_t^\alpha[w(x,y,t)] = \frac{d^\alpha w(x,y,t)}{dt^\alpha} = w^{(\alpha)}(x,y,t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{w(x,y,\tau)}{(t-\tau)^\alpha}d\tau \tag{9}$$

where c, c_α are rigidity coefficients–momentary and prolonged one [2], and α a rational number between 0 and 1, $0 < \alpha < 1$, [3, 5, 18].

In this paper we shall define sandwich structure as a system of material deformable bodies interconnected by *distrbuted light standard constrain elements* (*elastic, hereditary* or *creep*) and which are, in natural state, on defined interdistances (when distributed light constraint elements are unstressed). Sandwich structure is *ideally elastic* if it's material deformable bodies (plates, beams, belts) are pure elastic and bodies are interconnected by distributed light standard ideally elastic constraint elements [7, 10, 11, 18, 19, 20, 21]. Sandwich structure is a *standard hereditary disrete continuum* [3, 5, 18] if it's material deformable bodies (plates, beams, belts) are built of hereditary material and their bodies are interconnected by light standard hereditary elements. Sandwich structure is a *standard creep discrete continuum* [3, 14, 15, 16, 18] if its material deformable bodies (plates, beams, belts) are built of material with creeping properties and bodies are interconnected by light standard creep elements.

We shall define ***discrete-continuum homogeneous chain system*** [5, 18] as a system of equal material deformable bodies (plates, beams, belts) [3, 9, 14] which have same boundary contours and boundary conditions and can move transversaly to the characteristic «*elastic surface*» for plates $w_i(x, y, t), i = 1,2,...,M$ and «*elastic line*» for beams and belts $v_i(z,t), i = 1,2,...,N$ and which are interconnected by standard constraint light elements equal material constants. The chain is ideally elastic if all elements are ideally elastic [26]. The chain is standard hereditary if all elemets are built of hereditary material [2]. The chain is standard creep if if all elemets are built of material with creep properties [18, 5]. The number of degrees of freedom of each of these chains is equal M-infinity, where M is number of deformable bodies in chain, since we hypothesize that each deformable body can move transversaly to the characteristic «*elastic surface*» for plates, and «*elastic line*» for beams and belts, and which are interconnected by standard constraint light elements equal material constants.

3. SANDWICH STRUCTURE MODELS

3.1* ***Multiplate and multibeam sandwich system. Theoretical problem formulation and governing equations*** [3, 15]. Let us suppose that plates are thin and that it is not deplanation of the cross sections in the conditions of the creep material [8, 15]. Also, we suppose that always cross sections are orthogonal with respect to the middle surface (plane) of the plate. If thin plates are creep bent with small deflection, i.e., when the deflection of the middle surface is small compared with the thickness h, the same assumption can be made for both plates as in the Hedrih's papers [15, 3, 14].

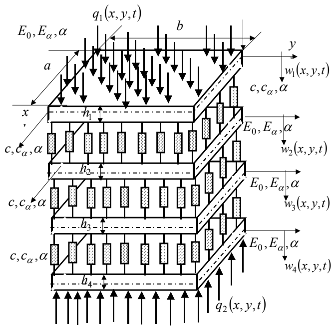
Now, let us consider finite number M isotropic, creeping, thin plates, width h_i, $i = 1,2,...,M$, modulus of elasticity E_i, Poisson's ratio μ_i and shear modulus G_i, plate mass distribution ρ_i. The plates are of constant thickness in the z-direction (see Fig. 1). The contours of the plates are parallel. Plates are interconnected by corresponding number $M-1$ creeping layers with the fractional order derivative constitutive relations type with constant surface stiffnesses. These creep-layers connected multiple plate system are a composite structure type, or sandwiched plates, or layered plates.

The origins of the corresponding number M coordinate systems are M corresponding sets at the corresponding centres in the nondeformed plates middle surfaces as shown in Fig. 1. and with parallel corresponding axes. The plates may be subjected to either a transversal distributed external loads $q_i(x,y,t)$, $i = 1,2,...,M$ along corresponding plates external surfaces. The problem at hand is to determine solutions.

The use of Love-Kirchhoff approximation make classical plate theory essentially a two dimensional phenomenon, in which the normal and transverse forces and bending and twisting moments on plate cross sections (see book by Rašković, 1965;[26]) can be found in terms of displacement $w_i(x, y, t)$, $i = 1,2,...,M$ of middle surface points, which is assumed to be a function of two coordinates, x and y and time t.

Let us denote with $\mathsf{D}_i = \dfrac{\mathbf{E}_i h^3}{12\left(1-\mu^2\right)}$, $\mathsf{D}_{i\alpha} = \dfrac{\mathbf{E}_{i\alpha} h^3}{12\left(1-\mu^2\right)}$, $i = 1,2,...,M$ corresponding bending cylindrical rigidity of creep plates. For homogeneous and isotropic plates material with parameters of material creep properties are equal $\alpha_x = \alpha_y = \alpha$; also, coefficients of rigidity of momentaneous and prolongeous one are: $\mathbf{E_{0x}} = \mathbf{E_{0}}_y = \mathbf{E_0}$ and $\mathbf{E}_{\alpha\mathbf{x}} = \mathbf{E}_{\alpha y} = \mathbf{E}_\alpha$ in all directions at corresponding point.. The coefficients of rigidity of momentaneous and prolongeous one for creep layer are c and c_α, and the parameter of layer material creep properties is $0 \le \alpha \le 1$.

Now, by using results of Hedrih, from papers [3, 14, 15], the relation (see Appendix B 1. from [15] and [8]) between stress components and strain components expressed by transversal displacements $w(x, y, t)$ of the plate middle surface corresponding point $N(x, y, 0)$ and coordinate z of the corresponding plate point $N(x, y, z)$. than we can write the following system of the M governing coupled partial fractional order differential equations of the creep connected multi plate system dynamics:



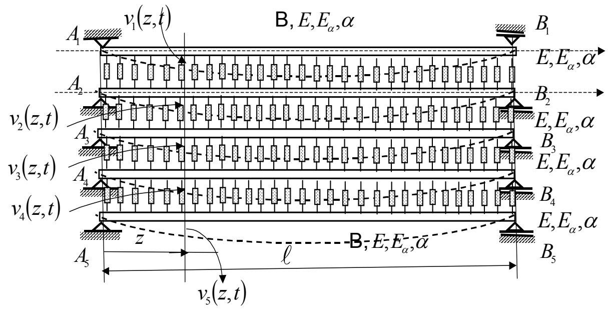


Figure 1. A creeping connected multi plate system **Figure 2.** A creeping connected multi beam system.

$$\frac{\partial^2 w_1(x,y,t)}{\partial t^2} + c_{(1)}^4 \left\{\left(1+\kappa_\alpha \mathsf{D}_t^\alpha\right)\left[\Delta\Delta w_{(1)}(x,y,t)\right]\right\} - a_{(1)}^2 \left\{\left(1+\kappa_\alpha^c \mathsf{D}_t^\alpha\right)\left[w_2(x,y,t) - w_1(x,y,t)\right]\right\} = \tilde{q}_1(x,y,t)$$

$$\frac{\partial^2 w_i(x,y,t)}{\partial t^2} + c_{(i)}^4 \left\{\left(1+\kappa_\alpha \mathsf{D}_t^\alpha\right)\left[\Delta\Delta w_i(x,y,t)\right]\right\} + a_{(i)}^2 \left\{\left(1+\kappa_\alpha^c \mathsf{D}_t^\alpha\right)\left[w_i(x,y,t) - w_{i-1}(x,y,t)\right]\right\} -$$
$$- a_{(i)}^2 \left\{\left(1+\kappa_\alpha^c \mathsf{D}_t^\alpha\right)\left[w_{i+1}(x,y,t) - w_i(x,y,t)\right]\right\} = -\tilde{q}_i(x,y,t)$$
$$i = 2,...,M-1 \tag{10}$$

$$\frac{\partial^2 w_M(x,y,t)}{\partial t^2} + c_{(M)}^4 \left\{\left(1+\kappa_\alpha \mathsf{D}_t^\alpha\right)\left[\Delta\Delta w_M(x,y,t)\right]\right\} + a_{(M)}^2 \left\{\left(1+\kappa_\alpha^c \mathsf{D}_t^\alpha\right)\left[w_M(x,y,t) - w_{M-1}(x,y,t)\right]\right\} = -\tilde{q}_M(x,y,t)$$

formulated in terms of M unknowns: the transversal displacement $w_i(x, y, t)$, $i = 1,2,...,M$ in direction of the axis z, of the plate middle surfaces (see Figure 1), where $\tilde{q}_i(x, y, t)$ loads and:

$c_{(i)}^4 = \dfrac{\mathsf{D}_i}{\rho_i h_i} = \dfrac{\mathbf{E}_0 h^3}{12\rho h\left(1-\mu^2\right)} = c_0^4$, $\kappa_\alpha = \dfrac{\mathbf{E}_\alpha}{\mathbf{E}_0}$, $\kappa_\alpha^c = \dfrac{c_\alpha}{c}$, $a_{(i)}^2 = \dfrac{c}{\rho h} = a_0^2$, $i = 1,2,...,M$.

The solutions of the governing system of corresponding coupled partial fractional order differential equations (10), we take in the eigen amplitude functions $\mathbf{W}_{(i)nm}(x,y)$, $i = 1,2,...,M$, $n,m = 1,2,3,4,....\infty$ expansions, from solution of the basic problem with decoupled equations [3, 14, 15] and with time coefficients in the form of unknown time functions $T_{(i)nm}(t)$, $n,m = 1,2,3,4,....\infty$, $i = 1,2,...,M$ describing their time evolution:

$$w_i(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \mathrm{W}_{(1)nm}(x,y)\mathrm{T}_{(i)nm}(t),\ i = 1,2,...,M \tag{11}$$

Than after introducing the (11) into the governing system of coupled partial fractional order differential equations for free and also for forced double plates oscillations (10) and by multiplying first and second equation with $\mathrm{W}_{(i)sr}(x,y)dxdy$ and after integrating along all surface of the plate middle surface and taking into account orthogonality conditions [15] and corresponding equal boundary conditions of the plates, we obtain the mn-family of systems containing coupled only M-ordinary fractional order differential equations for determination of the unknown time functions $T_{(i)nm}(t)$, $i = 1,2,...,M$, $n,m = 1,2,3,4,....\infty$ in the following form:

$$\begin{aligned}
&\ddot{\mathrm{T}}_{(1)nm}(t) + \omega_{(1)nm}^2\left(1 + \widetilde{\kappa}_{\alpha(1)nm}\mathsf{D}_t^\alpha\right)\mathrm{T}_{(1)nm}(t) - \left(a_{(1)}^2 + a_{(1)\alpha\, nm}^2\mathsf{D}_t^\alpha\right)\mathrm{T}_{(2)nm}(t) = f_{(1)nm}(t)\\
&\ddot{\mathrm{T}}_{(i)nm}(t) + 2\omega_{(i)nm}^2\left(1 + \widetilde{\kappa}_{\alpha(i)nm}\mathsf{D}_t^\alpha\right)\mathrm{T}_{(i)nm}(t) - \left(a_{(i)}^2 + a_{(i)\alpha\, nm}^2\mathsf{D}_t^\alpha\right)\left[\mathrm{T}_{(i-1)nm}(t) + \mathrm{T}_{(i+1)nm}(t)\right] = f_{(i)nm}(t)\\
&\qquad\qquad\qquad\qquad i = 2,...,M-1 \qquad\qquad n,m = 1,2,3,4,....\infty\\
&\ddot{\mathrm{T}}_{(M)nm}(t) + \omega_{(M)nm}^2\left(1 + \widetilde{\kappa}_{\alpha(M)nm}\mathsf{D}_t^\alpha\right)\mathrm{T}_{(M)nm}(t) - \left(a_{(M)}^2 + a_{(M)\alpha\, nm}^2\mathsf{D}_t^\alpha\right)\mathrm{T}_{(M-1)nm}(t) = -f_{(M)nm}(t);
\end{aligned} \tag{12}$$

where time known function $f_{(i)nm}(t)$, $i = 1,2,...,M$ are defined by following expressions:

$$f_{(i)nm}(t) = \frac{\int_0^a\int_0^b \widetilde{q}_i(x,y,t)\mathrm{W}_{(i)nm}(x,y)dxdy}{\int_0^a\int_0^b \left[\mathrm{W}_{(i)nm}(x,y)\right]^2 dxdy}\ . \tag{13}$$

The system of coupled fractional order differential equations (13) on unknown time-functions $T_{(i)nm}(t)$, $i = 1,2,...,M$, $n,m = 1,2,3,4,....\infty$, can be solved applying Laplace transforms [15].

For the case of the multibeam sandwich system [16, 21] we obtain similar system of coupled fractional order differential equations as (12), but with unknown time-functions $T_{(i)s}(t)$, $i = 1,2,...,M$, $s = 1,2,3,4,....\infty$ for every s-family of systems containing coupled only M-ordinary fractional order differential equations.

3.2* ***Multibelt sandwich system. Theoretical Problem Formulation and Governing Equations*** [11]. The sandwich belt system contain two belts constrained by distributed discrete light neglected mass belts with stiffness c . Both belts are represented by area of constant cross sections A along length ℓ between rolling and fixed bearings A and B , and by ρ the density of the belt material. Let us suppose that sandwich double belt

system is moving in the direction x with an axial velocity $v(t)$. Transversal vibrations of sandwich double belts are represented by transverse displacements $w_1(x,t)$ of lower belt and $w_2(x,t)$ of upper belt. Also, let us suppose that displacement is small, and that cross sections during the transverse vibration have no deplanations. Also, if we suppose that both belts are loaded by active axial force, due to the belts' tension, than in stressed state in the cross section appears normal stress with intensity σ, almost surely of constant intensity during the time vibrations and along the length of sandwich belt between bearings. Than we can conclude that normal stress σ in strings of sandwich double string system for a cross section during vibrations change only direction.
Let us introduce the following partial differential operator: $\mathbf{L}_{x,t}[\bullet]$

$$\mathbf{L}_{x,t}[\bullet]=\frac{\partial^2}{\partial t^2}-\left(c_0^2-v_0^2\right)\frac{\partial^2}{\partial x^2}+2v_0\frac{\partial^2}{\partial x\partial t}+2\delta v_0\frac{\partial}{\partial x}+2\delta\frac{\partial}{\partial t}+\kappa^2 \tag{14}$$

and governing partial differential equations are obtained the following form:

$$\mathbf{L}_{x,t}[w_1(x,t)]-\kappa^2 w_2(x,t)=0 \qquad \mathbf{L}_{x,t}[w_2(x,t)]-\kappa^2 w_1(x,t)=0 \tag{15}$$

where $c_0=\sqrt{\frac{\sigma}{\rho}}$, $\kappa=\sqrt{\frac{c}{\rho A}}$, $2\delta=\frac{b}{\rho A}$. By using independent coordinates in the following form: $\xi=x$, $\eta=\frac{v_0}{c_0^2-v_0^2}x+t$ and by transforming governing partial differential equations we obtain these equations in the simples form suitable for solution obtain.

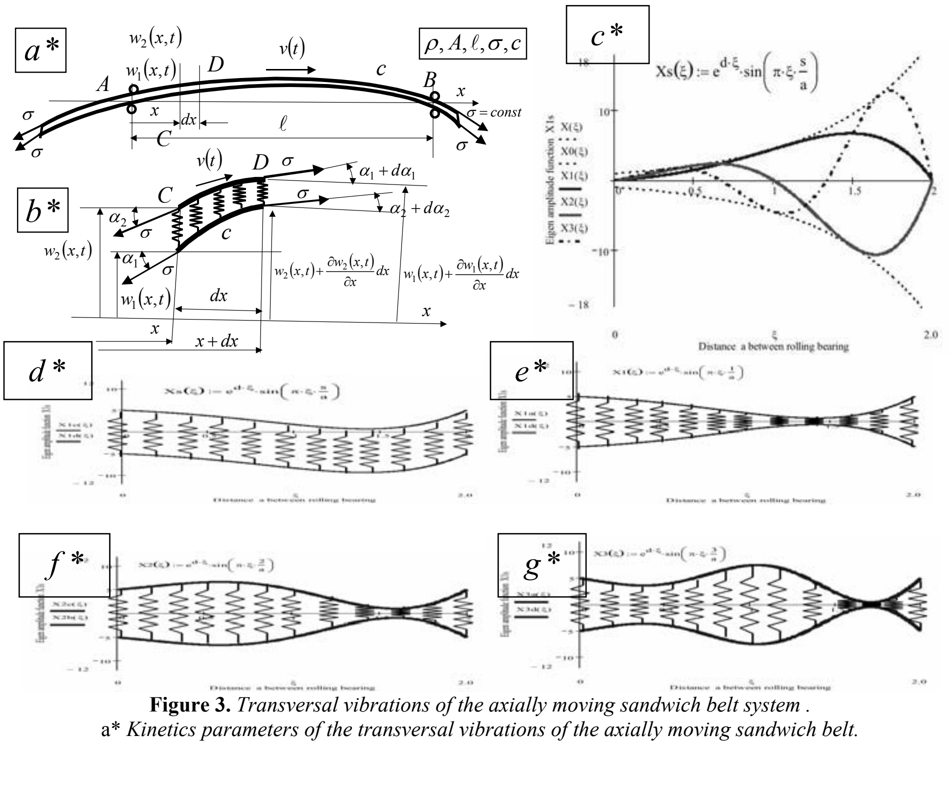


Figure 3. *Transversal vibrations of the axially moving sandwich belt system .*
a* *Kinetics parameters of the transversal vibrations of the axially moving sandwich belt.*

b* *Elementary segment of the axially moving sandwich belt with length* dx *and notations of the kinetics parameters;* **c*** *Eigen amplitude function for first three modes of the double belt system vibrations, Amplitude forms for transversal vibrations of the axially moving double sandwich belt system for some of possible cases: for first (d*) and (e*), for second(f*) and for third (g*) mode.*

Solution of the previous partial differential equation (15) can be looked for using Bernoulli's method of particular integrals in the form of multiplication of two functions [26, 22], from which the first $\mathbf{X}_{(i)}(\xi)$, $i=1,2$ depends only on space coordinate ξ and the second $\mathbf{Y}_{(i)}(\eta)$, $i=1,2$ is function of η:

$$w_{(i)}(\xi,\eta)=\mathbf{X}_{(i)}(\xi)\mathbf{Y}_{(i)}(\eta),\quad i=1,2 \tag{16}$$

and after denotation in the forms: $\tilde{\omega}^2=k^2\frac{c_0^2-v_0^2}{c_0^2}$, $\tilde{\delta}=\frac{\delta v_0}{\left(c_0^2-v_0^2\right)}$, $\tilde{\lambda}^2=\frac{k^2-\kappa^2}{\left(c_0^2-v_0^2\right)}$ we obtain:

$$\tilde{\mathbf{L}}_\eta\left[\mathbf{Y}_{(i)}(\eta)\right]+\tilde{\omega}^2\mathbf{Y}_{(i)}(\eta)=0 \quad \text{and} \quad \tilde{\mathbf{L}}_\xi\left[\mathbf{X}_{(i)}(\xi)\right]+\left[\tilde{\lambda}^2+\frac{\kappa^2}{\left(c_0^2-v_0^2\right)}\right]\mathbf{X}_{(i)}(\xi)=0 \tag{17}$$

and general and particular solutions must satisfy the boundary conditions: displacements in the rolling bearing must be equal to zero. And final one set of possible solution sets is:

$$\mathbf{X}_{(i)s}(x)=e^{\tilde{\delta}x}\sin\frac{s\pi}{\ell}x \qquad \text{where} \qquad \tilde{\delta}=\frac{\delta v_0}{\left(c_0^2-v_0^2\right)} \tag{18}$$

$$\mathbf{Y}_{(i)s}(x,t)=e^{-\delta\left(\frac{v_0}{c_0^2-v_0^2}x+t\right)}\left[A_s\cos q_s\left(\frac{v_0}{c_0^2-v_0^2}x+t\right)+B_s\sin q_s\left(\frac{v_0}{c_0^2-v_0^2}x+t\right)\right] \tag{19}$$

where

$$q_{(1,2)s}=\mp\sqrt{\left(\frac{s\pi}{\ell}\right)^2\frac{\left(c_0^2-v_0^2\right)^2}{c_0^2}+\left(\kappa^2-\delta^2\right)\frac{c_0^2-v_0^2}{c_0^2}}\,, \qquad s=1,2,...... \tag{20}$$

4. CONCLUDING REMARKS

The M coupled partial fractional order differential equations of transversal vibrations of a creeping connected multi plate system have been derived and presented by using author's results from Refs. [3-20]. Also, the M coupled partial fractional order differential equations of transversal vibrations of a creeping connected multi beam system have been presented in the light of mathematical analogy.

The analytical solutions of a system of M coupled partial fractional order differential equations of corresponding dynamical free and forced processes are obtained by using classical method of Bernoulli's particular integral and Laplace transform method. By using trigonometric method [26, 17, 2] and solution of the obtained system algebra equations with respect to $\mathsf{L}\left\{\mathbf{T}_{(i)nm}(t)\right\}$, $i=2,...,M-1$; $n,m=1,2,3,4,....\infty$. we obtain the determinant $\Delta_{nm}(p)$ of the nm-family of the system equations obtained by Laplace transform of the system equations (12).

For analysis, we can compare Laplace transform for the case of coupled double plates and for uncoupled plates for creep system and case for an ideal elastic system when $\alpha=0$, and we can conclude the following: It is shown that two-frequency-like regime

for free vibrations induced by initial conditions of double plate system corresponds to one mode vibrations. Analytical solutions show us that creeping connection between plates in M -multi plate system caused the appearance of similar M -frequency regime of the time function correspondent to one eigen amplitude function of one mode, and also that time functions of different the mn -family vibration modes $n,m = 1,2,3,4,....\infty$ are uncoupled. It is shown for every shape of vibrations. It is proved that in one of the mn -family vibration modes $n,m = 1,2,3,4,....\infty$ of the all M -creep connected plates are present M possibilities for appearance of the resonance-like dynamical states, and also for appearance of the dynamical absorption-like. And, at the end of this part, the analogy between mathematical descriptions of the discparate systems with plates and beams as in Refs. [2, 16, 17] is possible to be described by same type of the system equations. Also, the phenomenological mapping between dynamical processes is possible to identify.
And, at the end of the paper, if we compare the expressions for coupled and uncoupled two belts, we can conclude that for uncoupled belts' vibrations contain one frequency damped vibrations in one own amplitude shape, and for coupled vibrations contain two frequency damped vibrations in every one amplitude shape, and that these two-frequency dumped vibrations are uncoupled with relation of the other shape own vibrations. This is visible form expressions (17-19), as well as from following s -th particular solution:

$$w_{(i)s}(x,t) = e^{-\delta\left(\frac{v_0}{c_0^2 - v_0^2}x + t\right) + \tilde{\delta}x} \sum_{s=1}^{s=\infty} \sin\frac{s\pi}{\ell}x \left[R_s \cos\left\{ \sqrt{\left(\frac{s\pi}{\ell}\right)^2 \frac{\left(c_0^2 - v_0^2\right)^2}{c_0^2} + \left(\kappa^2 - \delta^2\right)\frac{c_0^2 - v_0^2}{c_0^2}} \left(\frac{v_0}{c_0^2 - v_0^2}x + t\right) + \beta_s \right\} \right]$$

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DINAMIKA SENDVIČ STRUKTURA

Rezime: *Dat je pregled modela i dinamike oscilovanja sendvič struktura sastavljenih od greda, ploča i traka od materijala različitih svojstava. Prikazne su konstitutivne relacije materijala elementa strukture: sa svojstvima elastičnosti, viskoelastičcnosti i puzanja. Dat je matematički opis dinamike sendvič struktura. Ukazuje se na svojstva oscilacija i vibracionih viskoelastičnih procesa. Data je vizuelizacija pojedinih modova procesa.*
Ključne reči: *Dinamika, sendvič struktura, ploča, greda, traka, viskoelastičnost, relaksaciono jezgro, rezolventa, izvodi necelog reda, analiza strukture vibracionih procesa.*

On the conditions for the existence of a plane of symmetry for anisotropic elastic material

Jovo P. Jarić [1]

Abstract: *We consider the problem of determining necessary and sufficient conditions for the existence of symmetry planes of an anisotropic elastic material. These conditions are given in several equivalent forms, and are used to determine special coordinate systems where the number of non-zero components in the elasticity tensor is minimized. By the method presented here it is also shown that an elastic solid has at least six coordinate systems with respect to which there are only 18 non-zero elastic constants and cannot possess more then ten traditional and distinct symmetrics by planes of symmetry.*

1 Introduction

This work is mainly motivated by the papers by Cowin and Mehrabadi [1] and by Norris [2]. We cite those results of there papers which are related to the results derived here. That also makes this paper self-contained.

In their extended paper [1] Cowin and Mehrabadi considered the problem of determining the material symmetry of an anisotropic elastic material. The main results of paper is based on the following
theorem 1: *The conditions*

$$c_{irpq}a_ra_pa_q = (c_{rspq}a_ra_sa_pa_q)a_i \tag{1.1}$$
$$c_{ikkj}a_j = (c_{pkkq}a_pa_q)a_i \tag{1.2}$$
$$c_{ijkk}a_j = (c_{pqkk}a_pa_q)a_i \tag{1.3}$$

and

$$c_{ijkm}b_jb_ka_m = (c_{rspq}b_sb_pa_ra_q)a_i \tag{1.4}$$

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consider a set of necessary and sufficient conditions for the vector **a** *to be the normal to a plane of symmetry of a material of given elasticities* c_{ijkm}. *The vector* **b** *is any vector perpendicular to* **a**. [2]

Here c_{ijkm} are Cartesian components of *the Hook's elasticity t e n s o r* for a homogeneous elastic solids which possesses the symmetries

$$c_{ijkm} = c_{jikm} = c_{ijmk} = c_{kmij}. \tag{1.5}$$

To prove sufficiency of the theorem, it was shown that if **a** and **b** are solutions of (1.1-1.4), than **a** is normal to a plane of symmetry and, with no loss of generality, one can take the coordinate axes x_1 and x_2 along **a** and **b**, respectively. With respect to such a coordinate system (1.2-1.4) yield

$$\begin{array}{ll} c_{i111} = c_{1111}\delta_{i1} & c_{ikk1} = c_{1kk1}\delta_{i1} \\ c_{i111} = c_{1111}\delta_{i1} & c_{i221} = c_{1221}\delta_{i1} \end{array} \tag{1.6}$$

or

$$c_{1112} = c_{1113} = c_{2212} = c_{2213} = c_{2321} = c_{2331} = c_{3312} = c_{3313} = 0. \tag{1.7}$$

The conditions (1.6) are requirements for monoclinic material symmetry where the x_1–coordinate direction is the normal to the plane of symmetry. Than, they concluded that any solution of (1.1-1.4) is the normal to a plane of symmetry.

The same problem was considerd by Norris [2] and the same theorem was stated in the following simplified version, due to Cowin, ([3], [4]),
theorem 2: *The necessary and sufficient conditions that the direction* **a** *be normal to a plane of symmetry are*

$$c_{irpq}a_r a_p a_q = (c_{rspq}a_r a_s a_p a_q)a_i \tag{1.1}$$

$$c_{ijkm}b_j b_k a_m = (c_{rspq}b_s b_p a_r a_q)a_i \tag{1.2}$$

for all direction **b** *perpendicular to* **a**.

These two theorems differ in conditions (1.2) and (1.3), which are, according to Norris, consequences of (1.1) and (1.4). From a mathematical point of view it means that the conditions (1.2) and (1.3) are redundant. To prove that (1.3) follows from (1.1) and (1.4) Norris [2] made use of (2.7) and write

$$\begin{aligned} c_{ijkk}a_j b_{\alpha i} &= c_{ijkl}a_j(a_k a_l + b_{\beta k}b_{\beta l})b_{\alpha i} \\ &= c_{ijkl}a_j a_k a_l + c_{ijkl}b_{\alpha i}b_{2k}b_{2l}a_j + c_{ijkl}b_{\alpha i}b_{3k}b_{3l}a_j. \end{aligned}$$

Then the first term on the right-hand side of this expression vanishes using (1.1) and, according to Norris, the secodn and the third terms are also zero by virtue of (1.4), what is not so obvious. Therefore $c_{ijkk}a_j$ is perpendicular to $\mathbf{b}_\alpha$, $(\alpha = 2, 3)$, so it must be parallel to **a** and (1.3) follows.

[2]Here and throughout the paper we used the summation convention for repeted indices.

Because of these and of the importance of problem considered, particularly from practical point of view, we belive that is worthwhile to examine it from a different angle. In fact, making use of the decomposition of a tensor in a coordinate system define by the orthonormal vectors **a**, $\mathbf{b}_\alpha$, $(\alpha = 2, 3)$, we derive exactly eight relations

$$\begin{aligned} c_\alpha &= c_{ijkl} b_{\alpha i} a_j a_k a_l = 0, \\ c_{\alpha\beta\gamma} &= c_{\beta\alpha\gamma} = c_{ijkl} b_{\alpha i} b_{\beta j} b_{\gamma k} a_l = 0 \end{aligned} \tag{a}$$

which constitute a set of necessary and sufficient condition for the vector **a** to be normal to a plane of symmetry of given elasticity tensor. We then proceed to derive these conditions in several equivalent forms. The conditions (a) enable us to define also special coordinate system where the number of non-zero components in the elasticity tensor is minimized. In addition to it we were able to prove the theorem which states that an elastic solid has at least six coordinate systems with respect to which there are only 18 non-zero elastic components. This represents the extension of Norris' result [2], who proved the existance of at least three such coordinate systems.

Further, under the assumption that the body possesses more than one plane of elastic symmetry, it is shown, by the use of the conditions (a), that a material cannot possess more than ten traditional and distinct symmetries by planes of symmetry, the result which is well known.

2 Preliminary definitions and theorems

Let **R** be a symmetric improper orthogonal tensor representing the reflection in a plane whose unit normal is **a**. Then [3],

$$R_{ij} = \delta_{ij} - 2a_i a_j, \tag{2.1}$$

$$R_{ij} = R_{ji}, \quad R_{ik} R_{kj} = R_{ki} R_{kj} = \delta_{ij}. \tag{2.2}$$

Moreover,

$$R_{ij} a_j = -a_i, \tag{2.3}$$

$$R_{ij} b_j = b_i, \tag{2.4}$$

for any vector **b** perpendicular to **a**, i.e.

$$\mathbf{a} \cdot \mathbf{b} = 0. \tag{2.5}$$

Let $\mathbf{b}_\alpha$ $(\alpha = 2, 3)$ be two unit orthogonal vectors each orthogonal to **a**:

$$b_{\alpha i} b_{\beta i} = \delta \alpha \beta, a_i b_{\alpha i} = 0. \tag{2.6}$$

From now on we refer to the vectors **a**, $\mathbf{b}_\alpha$ $(\alpha = 2, 3)$ as the new basis. In this basis the following identity holds:

$$\delta_{ij} = a_i a_j + b_{\alpha i} b_{\alpha j}. \tag{2.7}$$

Let h_i be Cartesian components of a vestor $\mathbf{h}$. Then

$$h_i = ha_i + H_\alpha b_{\alpha i} \tag{2.8}$$

where

$$h = h_i a_i, \quad H_\alpha = h_i b_{\alpha i}. \tag{2.9}$$

Lemma:

$$\begin{aligned}
c_{ijkl} = {} & ca_i a_j a_k a_l + \\
& + c_{1\alpha} b_{\alpha i} a_j a_k a_l + c_{2\alpha} a_i b_{\alpha j} a_k a_l + c_{3\alpha} a_i a_j b_{\alpha k} a_l + c_{4\alpha} a_i a_j a_k b_{\alpha l} + \\
& + d_{1\alpha\beta} b_{\alpha i} b_{\beta j} a_k a_l + d_{2\alpha\beta} b_{\alpha i} a_j b_{\beta k} a_l + d_{3\alpha\beta} b_{\alpha i} a_j a_k b_{\beta l} + d_{4\alpha\beta} a_i b_{\alpha j} a_j b_{\beta k} a_l + \\
& + d_{5\alpha\beta} a_i b_{\alpha j} a_k b_{\beta l} + d_{6\alpha\beta} a_i a_j b_{\alpha k} a_j b_{\beta l} + \\
& + c_{1\alpha\beta\gamma} b_{\alpha i} b_{\beta j} b_{\gamma k} a_l + c_{2\alpha\beta\gamma} b_{\alpha i} b_{\beta j} a_k b_{\gamma l} + c_{3\alpha\beta\gamma} b_{\alpha i} a_j b_{\beta k} b_{\gamma l} + c_{4\alpha\beta\gamma} a_i b_{\alpha j} b_{\beta k} b_{\gamma l} + \\
& + e_{\alpha\beta\gamma\delta} b_{\alpha i} b_{\beta j} b_{\gamma k} b_{\delta l}
\end{aligned}$$

for any tensor of fourth order.

This representation is unique and follows immediately by making use of (2.8) for indices i, j, k and l of the tensor c_{ijkl}. The expresions for the coefficients $c, c_{1\alpha}, \ldots, e_{\alpha\beta\gamma\delta}$ may be derived easily from (2.9) and (2.6).

For the elasticity tensor c_{ijkl} we have the following
theorem 3:

$$\begin{aligned}
c_{ijkl} = {} & ca_i a_j a_k a_l + \\
& + c_\alpha (b_{\alpha i} a_j a_k a_l + a_i b_{\alpha j} a_k a_l + a_i a_j b_{\alpha k} a_l + a_i a_j a_k b_{\alpha l}) + \\
& + c_{\alpha\beta} (b_{\alpha i} b_{\beta j} a_k a_l + a_i a_j b_{\alpha k} b_{\beta l}) + \\
& + d_{\alpha\beta} (b_{\alpha i} a_j b_{\beta k} a_l + b_{\alpha i} a_j a_k b_{\beta l} + a_i b_{\alpha j} b_{\beta k} a_l + a_i b_{\alpha j} a_k b_{\beta l}) + \\
& + c_{\alpha\beta\gamma} (b_{\alpha i} b_{\beta j} b_{\gamma k} a_l + b_{\alpha i} b_{\beta j} a_k b_{\gamma l} + b_{\alpha i} a_j b_{\beta k} b_{\gamma l} + a_i b_{\alpha j} b_{\beta k} b_{\gamma l}) + \\
& + e_{\alpha\beta\gamma\delta} b_{\alpha i} b_{\beta j} b_{\gamma k} b_{\delta l},
\end{aligned} \tag{2.10}$$

$$\begin{gathered}
c_{\alpha\beta} = c_{\beta\alpha}, \quad d_{\alpha\beta} = d_{\beta\alpha}, \quad c_{\alpha\beta\gamma} = c_{\beta\alpha\gamma}, \\
e_{\alpha\beta\gamma\delta} = e_{\beta\alpha\gamma\delta} = e_{\alpha\beta\delta\gamma} = e_{\gamma\delta\alpha\beta}.
\end{gathered} \tag{2.11}$$

Proof: From the Lemma and (1.5) we have

$$\begin{gathered}
c_{1\alpha} = c_{2\alpha} = c_{3\alpha} = c_{4\alpha} \equiv c_\alpha, \\
d_{1\alpha\beta} = d_{2\alpha\beta} = d_{6\alpha\beta} = d_{6\beta\alpha} \equiv c_{\alpha\beta} = c_{\beta\alpha}, \\
d_{2\alpha\beta} = d_{3\beta\alpha} = d_{4\alpha\beta} = d_{5\beta\alpha} \equiv d_{\alpha\beta} = d_{\beta\alpha},
\end{gathered}$$

$$\begin{gathered}
c_{1\alpha\beta\gamma} = c_{1\beta\alpha\gamma} = c_{2\alpha\beta\gamma} = c_{2\beta\alpha\gamma}, \\
c_{3\alpha\beta\gamma} = c_{3\alpha\gamma\beta} = c_{4\alpha\beta\gamma} = c_{4\alpha\gamma\beta}, \\
c_{1\alpha\beta\gamma} = c_{3\gamma\alpha\beta} \equiv c_{\alpha\beta\gamma} = c_{\beta\alpha\gamma}, \\
e_{\alpha\beta\gamma\delta} = e_{\beta\alpha\gamma\delta} = e_{\alpha\beta\delta\gamma} = e_{\gamma\delta\alpha\beta}. \Box
\end{gathered}$$

From (2.10) and (2.6) we may write the expressions for all coefficients in (2.10). For later purpose, we write

$$\begin{aligned} c_\alpha &= c_{ijkl} b_{\alpha i} a_j a_k a_l, \\ c_{\alpha\beta\gamma} &= c_{ijkl} b_{\alpha i} b_{\beta j} b_{\gamma k} a_l. \end{aligned} \tag{2.12}$$

Now we are ready to prove
theorem 4:

$$c_\alpha = c_{ijkl} b_{\alpha i} a_j a_k a_l = 0 \tag{2.13}$$

$$c_{\alpha\beta\gamma} = c_{ijkl} b_{\alpha i} b_{\beta j} b_{\gamma k} a_l = 0 \tag{2.14}$$

constitute a set of necessery and sufficient conditions for the vestor **a** *to be the normal to a plane of symmetry of given elasticities* c_{ijkl}.

P r o o f: The conditions are necessary. Indeed, if **R** belongs to the symmetry group of the material whose elasticity tensor is c_{ijkl} then one must have

$$c_{ijkl} = R_{ip} R_{jq} R_{kr} R_{ls} c_{pqrs.} \tag{2.15}$$

But, from (2.15), (2.16), (2.10), (2.3) and (2.4) at once follows (2.13) and (2.14). Then we may write

$$\begin{aligned} c_{ijkl} = {} & c a_i a_j a_k a_l + c_{\alpha\beta}(b_{\alpha i} b_{\beta j} a_k a_l + a_i a_j b_{\alpha k} b_{\beta l}) + \\ & + d_{\alpha\beta}(b_{\alpha i} a_j b_{\beta k} a_l + b_{\alpha i} a_j a_k b_{\beta l} + a_i b_{\alpha j} b_{\beta k} a_l + a_i b_{\alpha j} a_k b_{\beta l}) + \\ & + e_{\alpha\beta\gamma\delta} b_{\alpha i} b_{\beta j} b_{\alpha k} b_{\beta l}. \end{aligned} \tag{2.16}$$

Second, the conditions are sufficient. Indeed, because of (2.13) and (2.14) the relation (2.16) holds, and from (2.16), (2.3) and (2.4) it follows that

$$R_{ip} R_{jq} R_{kr} R_{ls} c_{pqrs} = c_{ijkl}.$$

From (2.15) we conclude that **R** defined by **a** belongs to the symmetry group. This means that **a**, satisfying (2.13) and (2.14), is normal to a plane of symmetry. □

Since $c_{\alpha\beta\gamma} = c_{\beta\alpha\gamma}$ the number of conditions (2.13) and (2.14) is **eight**. In a case considered by Cowin et al. [1], when

$$a_i = \delta_{i1}, \quad b_{\alpha i} = \delta_{\alpha i},$$

we may write

$$c_{\alpha 111} = 0, \quad c_{\alpha\beta\gamma 1} = 0 \tag{2.17}$$

or, equivalently,

$$c_{2111} = c_{3111} = c_{2221} = c_{2321} = c_{2331} = c_{3321} = c_{3331} = 0. \tag{2.18}$$

Of course (2.18) and (2.17) are identical become of (2.15).

3 Equivalent forms of necessary and sufficient conditions for a plane symmetry

The conditions (2.13) and (2.14) can be written in equivavelent forms.
I.

$$
\begin{aligned}
i) \qquad c_{ijkl}a_j a_k a_l &= (c_{pqrs}a_p a_q a_r a_s)a_i & (1.1)\\
c_{ijkl}b_{\alpha i}b_{\beta k}a_l &= (c_{pqrs}a_p b_{\alpha q}b_{\beta r}a_s)a_i & (3.1)
\end{aligned}
$$

or

$$
\begin{aligned}
i) \qquad c_{ijkl}a_j a_k a_l &= (c_{pqrs}a_p a_q a_r a_s)a_i & (1.1)\\
c_{ijkl}a_j b_{\alpha k}b_{\beta l} &= (c_{pqrs}a_p a_q b_{\alpha r}b_{\beta s})a_i & (3.2)
\end{aligned}
$$

are equivalent to (2.13) and (2.14).

Indeed (1.1) may be derived from (2.13) when it is multiplied by e.g. $b_{\alpha m}$, and (2.7) is used. In the same way (3.1) follows from (2.14) and (2.7) when (2.14) is multiplied by $b_{\alpha m}$. The same procedure may be applied for a derivation of (3.2).

II. The conditions (2.13) and (2.14), or equivalenty $i)$ and $ii)$, may be written as

$$
\begin{aligned}
iii) \qquad c_{ijkl}a_j a_k a_l &= (c_{pqrs}a_p a_q a_r a_s)a_i & (1.1)\\
c_{ijkl}b_j b_k a_l &= (c_{pqrs}a_p b_q b_r a_s)a_i & (3.3)\\
iv) \qquad c_{ijkl}a_j a_k a_l &= (c_{pqrs}a_p a_q a_r a_s)a_i & (1.1)\\
c_{ijkl}a_j b_k b_l &= (c_{pqrs}a_p a_q b_r b_s)a_i & (3.4)
\end{aligned}
$$

where **b** is any vector perpendicular to **a**.

Indeed, for any vector **b** in the plane defined by vectors $\mathbf{b}_\alpha$ $(\alpha = 2, 3)$ we have

$$b_i = \lambda_\alpha b_{\alpha i} \tag{3.5}$$

where for any vector $\lambda_\alpha$are arbitrary. Then multiplying (3.1) and (3.2) with $\lambda_\alpha\lambda_\beta$ we arrive at (3.3) and (3.4) respectively.

On the other hand, making use of (3.5) in (3.3) we obtain

$$c_{ijkl}b_{\alpha j}b_{\beta k}a_{l(\alpha,\beta)} = (c_{pqrs}a_p b_{\alpha q}b_{\beta r}a_s)a_i$$

since λ_α is arbitrary and $\lambda_\alpha\lambda_\beta$ is symmetric. Multiplying this by $b_{\gamma i}$ and taking into account $(2.6)_2$ and $(2.12)_2$ we obtain

$$c_{\alpha(\beta\gamma)} = 0 \quad \text{or} \quad c_{\alpha\beta\gamma} = -c_{\alpha\gamma\beta}$$

i.e. $c_{\alpha\beta\gamma}$ is symmetric with respect to the last two indices. Since by definition $c_{\alpha\beta\gamma}$ is also symmetric in the frst two indices it follows that must be zero. Indeed,

$$c_{\alpha\beta\gamma} = c_{\beta\alpha\gamma} = -c_{\beta\gamma\alpha} = -c_{\gamma\beta\alpha} = c_{\gamma\alpha\beta} = c_{\alpha\gamma\beta} = -c_{\alpha\beta\gamma},$$

i.e. (2.14) holds. Further, (2.13) follows from (1.1) when it is multiplied by $b_{\alpha i}$ and when $(2.6)_2$ is taking into account. In the same way may prove that (2.13) and (2.14) follows from (1.1) and (3.4) i.e. *iv*).
Remark 1. As a consequence of *iii*) and *iv*) it follows that the conditions (1.2) and (1.3) stated in the theorem 1 are redundant. The conditions (1.2) and (1.3) are equivalent to the conditions

$$c_{ijkl}b_{\alpha j}b_{\alpha k}a_l = (c_{pqrs}a_p b_{\alpha q}b_{\alpha r}a_s)a_i,$$
$$c_{ijkl}a_j b_{\alpha k}b_{\alpha l} = (c_{pqrs}a_p a_q b_{\alpha r}b_{\alpha s})a_i,$$

respectively. They may be derived from (1.2) and (1.3) by making use of (1.1) and (2.7). They are included into conditions (3.3) and (3.4) respectively.

III. The conditions (2.13) and (2.14) may be expressed only in terms of **a**. To show this we multiply (2.14) by, e.q., $b_{\alpha p}b_{\beta q}b_{\gamma r}$. Then making use of (2.7) and (1.1) we obtain, after rearranging the indices and the terms,

$$c_{ijkl}a_l - (c_{pjkl}a_i + c_{ipkl}a_k)a_p a_l + 2ca_i a_j a_k = 0. \tag{3.6}$$

Remark 2. In a case when material is isotropic (3.6) holds for any unit vector **a**.
Remark 3. The conditions (2.13) and (2.14) are expressed in orthonormal basis defined by $\mathbf{b}_\alpha$ $(\alpha = 2, 3)$. But they may be expressed in any other basis defined by two linearly independent vectors, say $\mathbf{g}_\alpha$ so that

$$g_{\alpha i} = h_{\alpha\beta}b_{\alpha i}, \quad \det(h_{\alpha\beta}) \neq 0. \tag{3.7}$$

Then

$$k_\alpha \equiv h_{\alpha\beta}c_\beta = c_{ijkl}g_{\alpha i}a_j a_k a_l \tag{3.8}$$
$$k_{\alpha\beta\gamma} \equiv h_{\alpha\lambda}h_{\beta\mu}h_{\gamma\nu}c_{\lambda\mu\nu} = c_{ijkl}g_{\alpha i}g_{\beta j}g_{\gamma k}a_l \tag{3.9}$$

are expressed in terms of $\mathbf{g}_\alpha$ so that insted of (2.13) and (2.14) we may write

$$k_\alpha = 0 \tag{3.10}$$
$$k_{\alpha\beta\gamma} = 0 \tag{3.11}$$

as a necessary and sufficient conditions for **a** to be a vector normal to a plane of symmetry.

4 Special coordinate systems

It is well known ([3], [4], [5]) that there exists a coordinate system in which a monoclinic solid has only 12 non-zero elastic moduli.

I. The representation of c_{ijkl} given by (2.16) enables one to find two such coordinate systems. From linear algebra [6], it is known that for any real symmetric matrix there exists a orthogonal system, defined by the eigenvectors of the matrix, in wich the symmetric matrix is diogonal.

To this ena we write

$$c_{\alpha\beta} = c_{ijkl} b_{\alpha i} b_{\beta j} a_k a_l \tag{4.1}$$

$$d_{\alpha\beta} = c_{ijkl} b_{\alpha i} a_j b_{\beta k} a_l. \tag{4.2}$$

We also make use of the fact that (2.16) holds for any orthonormal basis which constains **a**.

In the basis of its eigenvestors $c_{\alpha\beta}$ has the property that

$$c_{23} = 0 \quad \text{or} \quad c_{2311} = 0 \tag{4.3}$$

when we take $b_{\alpha i} = \delta_{\alpha i}$, $(\alpha = 2, 3)$ as the eigenvectors of $c_{\alpha\beta}$. By the same reasoning we have

$$d_{23} = 0 \quad \text{or} \quad c_{2131} = 0 \tag{4.4}$$

in the basis of eigenvectors of $d_{\alpha\beta}$.

Of course in each of these coordinate systems (2.13) and (2.14), or equivalently (1.7), hold. Then in coordinate systems defined by eigenvectors of $c_{\alpha\beta}$ and $d_{\alpha\beta}$, one has only 12 non-zero components of c_{ijkl}.

Remark 4. It may happen that these two coordinate systems coincide. In that case both (4.3) and (4.4) would hold simultaneouslt and we would have 11 non-zero components of c_{ijkl}. To be so it is necessary and sufficient that the matrices of the elements $c_{\alpha\beta}$ and $d_{\alpha\beta}$ commute [6].

II. Up to now we did not make use of the assumption that the strain energy function W has to be positive definite as a function of strain tensor E_{ij}. By definition

$$2W = c_{ijkl} E_{ij} E_{kl} > 0 \tag{4.5}$$

for any $E_{ij} \neq 0$. Since E_{ij} is symmetric it easy to show that

$$E_{ij} = e a_i a_j + e_\alpha (b_{\alpha i} a_j + b_{\alpha j} a_i) + e_{\alpha\beta} b_{\alpha i} b_{\alpha j} \tag{4.6}$$

in the orthonormal bases $(\mathbf{a}, \mathbf{b}_\alpha)$. Then using (4.6) and (2.16) in (4.5) we obtain

$$2W = ce^2 + 2c_{\alpha\beta} e_{\alpha\beta} e + 4d_{\alpha\beta} e_\alpha b_\beta + e_{\alpha\beta\gamma\delta} e_{\alpha\beta} e_{\gamma\delta} > 0. \tag{4.7}$$

Since (4.7) has to be satisfied for any e, e_α and $e_{\alpha\beta}$ it must be that

$$c > 0 \tag{4.8}$$

and

$$\varphi = d_{\alpha\beta} e_\alpha e_\beta > 0, \tag{4.9}$$

$$\omega = e_{\alpha\beta} e_\alpha e_\beta > 0, \tag{4.10}$$

i.e. φ and ω are positive definite forms of e_α and $e_{\alpha\beta}$ respectively.

We focus our attention on (4.10). To this ena we use (2.16) and (2.6) and write

$$e_{\alpha\beta\gamma\delta} = c_{ijkl} b_{\alpha i} b_{\beta j} b_{\gamma k} b_{\delta l}. \tag{4.11}$$

But from (2.11) we see that $e_{\alpha\beta\gamma\delta}$ and c_{ijkl} have the same symmetric properties. Moreover, from (4.5) and (4.10) W and ω are positive definite. Then we may apply Kolodner's approach [7] to show that there exist at least two distinct unit vectors ω for which

$$e_{\alpha\beta\gamma\delta} n_\beta n_\gamma n_\delta = \lambda n_\alpha. \tag{4.12}$$

To each of them corresponds a unit vector $\mathbf{m}$ such that

$$e_{\alpha\beta\gamma\delta} n_\beta n_\gamma n_\delta = \mu m_\alpha. \tag{4.13}$$

Of course $\mathbf{n}$ and $\mathbf{m}$ are in the plane of symmetry and $\mathbf{n}\cdot \mathbf{m} = 0$ since $e_{\alpha\beta\gamma\delta} n_\beta n_\gamma$ is symmetric. The vector $\mathbf{a}$ and the two vectors $\mathbf{n}$, $\mathbf{m}$ define two orthogonal bases. Further, substituting (4.11) into (4.12) we obtain

$$c_{ijkl} b_{\alpha i} N_j N_k N_l = \lambda n_\alpha. \tag{4.12_1}$$

where

$$N_i = n_\alpha b_{\alpha i} \tag{4.14}$$

represents $\mathbf{n}$ in the orthogonal basis defined by $\mathbf{b}_\alpha$ $(\alpha = 2, 3)$. Multiplying (4.12_1) by $b_{\alpha m}$ and using (4.14) and (2.7) we obtain

$$c_{ijkl} N_j N_k N_l - (c_{mjkl} a_m N_j N_k N_l) a_i = \lambda N_i.$$

But

$$c_{mjkl} a_m N_j N_k N_l = 0$$

taking into account that $\mathbf{a}\cdot \mathbf{n} = 0$, because of (4.14) and (2.6), and (3.3) holds for any vector perpendicular to $\mathbf{a}$. Finally we obtain

$$c_{ijkl} N_j N_k N_l = \lambda N_i. \tag{4.15}$$

Of course $\lambda > 0$ in consequence of positivity conditions which ensure the physical meaning of λ and (4.15). Indeed λ is related to the square of the magnitude ν of the velocity ν of longitudinal wave which propagates in $\mathbf{n}$ direction.

In the coordinate system determined by one set of vectors $\mathbf{a}$, $\mathbf{n}$ and $\mathbf{m}$ from (4.15) we have that

$$c_{ijkl} M_i N_j N_k N_l = 0, \quad M_i = m_\alpha b_{\alpha i}, \tag{4.16}$$

Without loss of generality we may identify $\mathbf{n}$ with $\mathbf{b}_\alpha$ and $\mathbf{m}$ with $\mathbf{b}_\beta$ $(\alpha \neq \beta)$. Moreover, for $b_{\alpha i} = \delta_{\alpha i}$ $(\alpha = 2, 3)$, from (4.16) we obtain

$$c_{\alpha\beta\beta\beta} = 0, \tag{4.17}$$

where we do not sum over indecs β. Hence for

$$\mathbf{n} = \mathbf{b}_2 : \quad c_{3222} = 0 \tag{4.18}$$

and for

$$\mathbf{n} = \mathbf{b}_3 : \quad c_{2333} = 0. \tag{4.19}$$

Then from (1.7), (4.15) and (4.18) or (4.19) we may conclude:

In a plane of symmetry there exist at list two dinstinct directions along which longitudinal waves will propagate. In the coordinate systems defined by these directions and the direction normal to the plane of symmetry the elasticity tensor has only 12 non-zero components.

Further, from (4.12) we see that $\mathbf{m}$ is an eigenvector of $e_{\alpha\beta\gamma\delta}n_\beta n_\gamma$ as well as of $e_{\alpha\beta\gamma\delta}n_\gamma n_\delta$, i.e. in addition to (4.13) it also satisfies the relations

$$e_{\alpha\beta\gamma\delta}m_\beta n_\gamma n_\delta = \tau m_\alpha. \tag{4.20}$$

Also from

$$c_{ijkl}a_j a_k a_l = c a_i \tag{1.1}$$

we see that $\mathbf{a}$ is an eigenvector of $c_{ijkl}a_j a_k$ as well as of $c_{ijkl}a_k a_l$. Both of them are symmetric but generally different. Then to each of them coressponds its set of orthonormal vectors, say ν_α and μ_α, $(\alpha = 2, 3)$, which are generally different. Norris [2] investigated only $c_{ijkl}a_j a_k$ with eigenvectors $\mathbf{a}$, ν_α and concluded correctly that in that basis

$$c_{2333} = c_{1333} = c_{1323} = 0 \tag{4.21}$$

under the assumption that $a_i = \delta_{3i}$, $\nu_{\alpha i} = \delta_{\alpha i}$.

But if we investigate

$$c_{ijkl}\mu_{\alpha j}a_k a_l = d\nu_{\alpha i}$$

we obtain also

$$c_{2333} = c_{1333} = c_{1223} = 0 \tag{4.22}$$

when we take $a_i = \delta_{3i}$, $\mu_{\alpha i} = \delta_{\alpha i}$. The conditions given by (4.21) and (4.22) generally differ. Then, having in mind that Kolodner [7] showed that for an anisotropic elastic body there are at least three directions which satisfy (1.1), we may summarized our results in **theorem 5:** *An elastic solid has at least six coordinate systems with respect to which there are only 18 non-zero elastic constants. If the solid possesses a plane of symmetry four of the coordinate systems have as a common direction the normal to the symmetry and the solid has 12 non-zero moduli when referred to these coordinate systems. Moreover, in the plane of symmetry there are at least two directions along which longitudinal waves will propagate.*

This theorem extends Norris' results stated in his **theorem 5:**

An elastic solid has at least three coordinate systems with respect to which there are only 18 non-zero elastic constants. If the solid possesses a plane of symmetry, three of the coordinate systems have as a common direction the normal to the plane of symmetry and the solid has 12 non-zero moduli when referred to these systems.

Remark 5: It may happen that two bases defined by **a**, **n** and **m** coincide. In that course, in addition to (4.15), we will have

$$c_{ijkl}M_jM_kM_l = \nu M_i \tag{4.23}$$

i.e. the directions of propagation of longitudinal waves in the plane of symmetry will be perpendicular to each other. Also (4.17) or (4.18) and (4.19) will hold simultaneously and in that bases the elasticity tensor will have only 11 non-zero components.

Remark 6: The condition (4.8) enables one to relate c to the squared speed of propagation of longitudinal waves in the **a** direction. Also, from (4.9) we conclude that $d_{\alpha\beta}$ is positive definite. Then, since $d_{\alpha\beta}$ and $c_{\alpha\beta}$ are symmetric, there exists a real nonsingular linear transformation [6], which defines (generally) a nonortogonal coordinate system in which $d_{\alpha\beta}$ and $c_{\alpha\beta}$ are diagonal. Since we consider here only orthogonal systems we omit further discussion of this possibility.

5 Two orthogonal planes of elastic symmetry

Let us assume that there is an other plane of elastic symmetry orthogonal to the plane of symmetry defined by the vector **a**. With no loss in generality one can take $\mathbf{b}_2$ as its unit normal vector. Then

$$S_{ij} = \delta_{ij} - 2b_{2i}b_{2j} \tag{5.1}$$

$$S_{ij} = S_{ji}, \quad S_{ik}S_{jk} = S_{ki}S_{kj} = \delta_{ij} \tag{5.2}$$

is a symmetric improper orthogonal tensor representing the reflection in the plan with unit normal $\mathbf{b}_2$. Also

$$S_{ij}a_j = a_i, \quad S_{ij}b_{3j} = b_{3i}, \quad S_{ij}b_{2j} = -b_{2i} \tag{5.3}$$

because of (5.1) and (2.6). Then from the conditions

$$c_{ijkl} = S_{ip}S_{jq}S_{kr}S_{ls}c_{pqrs} \tag{5.4}$$

and (4.1), (4.2), (4.11) and (5.3) we obtain

$$c_{23} = 0, \quad d_{23} = 0, \quad e_{2333} = 0, \quad e_{2223} = 0, \tag{5.5}$$

or, for $a_i = \delta_{1i}$, $b_{\alpha i}$,

$$c_{2311} = 0, \quad c_{2131} = 0, \quad c_{2333} = 0, \quad c_{2223} = 0. \tag{5.6}$$

From (2.13) and (2.14) and (5.5) or (1.7) and (5.6), we see that c_{ijkl} has 9 non-zero components. Also (2.16) reduces to

$$\begin{aligned} c_{ijkl} = {} & ca_ia_ja_ka_l+ \\ & + c_{\alpha\alpha}(b_{\alpha i}b_{\alpha j}a_ka_l + a_ia_jb_{\alpha k}b_{\alpha l})+ \\ & + d_{\alpha\alpha}(b_{\alpha i}a_jb_{\alpha k}a_l + b_{\alpha i}a_ja_kb_{\alpha l} + a_ib_{\alpha j}b_{\alpha k}a_l + a_ib_{\alpha j}a_ka_kb_{\alpha l})+ \\ & + e_{\alpha\beta\gamma\delta}b_{\alpha i}b_{\beta j}b_{\gamma k}b_{\delta l} \end{aligned} \tag{5.7}$$

taking into account (5.5).

Obviously (5.7) will not change if $\mathbf{a} \to \mathbf{a}$, $\mathbf{b}_2 \to \mathbf{b}_2$, $\mathbf{b}_3 \to -\mathbf{b}_3$. The orthogonal transformation which has this property if defined by

$$T_{ij} = \delta_{ij} - 2b_{3i}b_{3j}. \tag{5.8}$$

But then

$$c_{ijkl} = T_{ip}T_{jq}T_{kr}T_{ls}c_{pqrs} \tag{5.9}$$

and $\mathbf{T}$, defined by (5.8), belongs to the symmetry group of c_{ijkl}. Geometrically it represents the plane of material symmetry defined by its unit normal vector $\mathbf{b}_3$. Since $\mathbf{a}$ and $\mathbf{b}_\alpha$ represent orthonormal vectors we conclude:

If a material has two orthogonal planes of symmetry then the plane orthogonal to them is also a plane of symmetry. A material with three orthogonal planes of elastic symmetry has 9 non-zero components of c_{ijkl}. In a coordinate system defined by the intersections of the planes of symmetry the zero componants of c_{ijkl} are given by (1.7) and (5.6).

These results are not new. But making use of this conclusion, **Remark 4, Remark 5,** we indeed prove

Theorem 6:

A necessary and sufficient condition that a material which possesses a plane of symmetry has three orthogonal planes of elastic symmetry it that the matrices $c_{\alpha\beta}$ and $d_{\alpha\beta}$ commute and their eigenvectors define the directions along which longitudinal waves may propagate.

6 Further characterization of the distinct elastic symmetries by two or more planes of symmetry

We turn now to the case when there are two, generally, nonorthogonal planes of symmetry defined by their unit normal vectors $\mathbf{a}$ and $\mathbf{m}$. To simplify the investigation:

First, one can take the coordinate axes x_1 along the direction of $\mathbf{a}$, and x_3 along the direction normal to the plane defined by vector $\mathbf{a}$ and $\mathbf{m}$, so that

$$a_i = \delta_{1i} \tag{6.1}$$

$$m_i = \lambda_\alpha \delta_{\alpha i}, \quad (\alpha = 1, 2) \tag{6.2}$$

$$\lambda_a \lambda_\alpha = 1, \quad (\lambda_1 = \cos\Theta,\ \lambda_2 = \sin\Theta). \tag{6.3}$$

The unit vector $\mathbf{n}$, orthogonal to $\mathbf{m}$ and $\mathbf{e}_3$, the unit vector along x_3 axes, is in the symmetry plane of $\mathbf{m}$ so that

$$n_i = \mu_\alpha \delta_{\alpha i} \tag{6.4}$$

$$\mu_\alpha = e_{\beta\alpha}\lambda_\beta. \tag{6.5}$$

(Here and further $e_{\alpha\beta}$ represents a tensor of alternation). In addition to (1.7), according to (2.13) and (2.14), the following set of conditions must be satisfied

$$c_{ijkl}\left\{\begin{array}{c}\delta_{3i}\\ n_i\end{array}\right\}m_j m_k m_l = 0 \tag{6.6}$$

$$c_{ijkl}\left\{\begin{array}{c}\delta_{3i}\delta_{3j}\delta_{3k}\\ \delta_{3i}\delta_{3j}n_k\\ \delta_{3i}n_j\delta_{3k}\\ \delta_{3i}n_j n_k\\ n_i n_j\delta_{3k}\\ n_i n_j n_k\end{array}\right\}m_l = 0. \tag{6.7}$$

This set of equations reduces to

$$\begin{gathered}\lambda_2 c_{3332} = 0\\ \lambda_2 c_{1213} = \lambda_2 c_{1123} = -\lambda_2 c_{2223} = 0\\ \lambda_2(4\lambda_1^2 - 1)c_{2223} = 0\\ \lambda_2\lambda_1(c_{3311} - c_{3322}) = 0\\ \lambda_2\lambda_1(c_{1313} - c_{2323}) = 0\\ \lambda_1(2\lambda_2^2 - 1)[c_{1111} - (2c_{1212} + c_{1122})] = 0\\ \lambda_2\lambda_1(c_{1111} - c_{2222}) = 0.\end{gathered} \tag{6.8}$$

where we have used (6.3). On the bases of these equations we identify the following cases:

a) Obviously these equations are identically satisfied for $\lambda_2 = 0$. In that case $\mathbf{m} = \mathbf{a}$ and there is only one plane of symmetry, the case which was already discussed. – *Monoclinic symmetry.*

Further, we will assume that $\lambda_2 \neq 0$. Then the set of equations (6.8) becomes

$$\begin{gathered}c_{3332} = 0\\ c_{1213} = c_{1123} = -c_{2223} = 0\\ \lambda_1(c_{1111} - c_{2222}) = 0\\ \lambda_1(c_{3311} - c_{3322}) = 0\\ \lambda_1(c_{1313} - c_{2323}) = 0\\ (4\lambda_1^2 - 1)c_{2223} = 0\\ \lambda_1(2\lambda_2^2 - 1)[c_{1111} - (2c_{1212} + c_{1122})] = 0.\end{gathered} \tag{6.9}$$

b) Then we have the following cases:

i) $\lambda_1 = 0$

$$c_{1213} = c_{1123} = c_{2223} = c_{3332} = 0 \tag{5.6}$$

and we come to the case which also already was discussed, when there are three orthogonal planes of symmetry – *Rhombic symmetry.*

ii) $\lambda_1 \neq 0$.

Then in additional to $(6.9)_{1,2}$ we have

$$\begin{aligned} c_{1111} &= c_{2222} \\ c_{3311} &= c_{3322} \\ c_{1313} &= c_{2323} \end{aligned} \tag{6.10}$$

and

$$\begin{aligned} (4\lambda_1^2 - 1)c_{2223} &= 0 \\ (2\lambda_1^2 - 1)[c_{1111} - (2c_{1212} + c_{1122})] &= 0. \end{aligned} \tag{6.11}$$

ii)$_1$ $\lambda_1^2 = 1/2$ or $\Theta = \pm\pi/4$ – *Tetragonal symmetry*

Then (6.9) and (6.10) hold. From (1.7) and (5.6) we see that $\Theta = \pm\pi/2$ define also a plane of symmetry.

ii)$_2$ $\lambda_1^4 = 1/2$ or $\Theta = \pm\pi/3$ – *Trigonal symmetry*

Then $(6.9)_{1,2}$ and (6.10) hold and

$$c_{1212} = dfrac12(c_{1111} - c_{1122}). \tag{6.12}$$

c) For any λ_1 (or λ_2) – *Transverse isotropy - Hexagonal symmetry*

In this case (6.8) are satisfied for any value of λ_1 and λ_2 so that the plane defined by its unit vector $\mathbf{e}_3$ is a plane of isotropy. Moreover, the plane defined by $\mathbf{e}_3$, i.e. the plane of isotropy, is also a plane of symmetry. The body is transversely isotropic. Then, in addition to (1.7), (5.6), (6.10) and (6.12) hold. Particulary, it also holds for $\lambda_2 \neq 0$, $\lambda_1 \neq 0$, $2\lambda_1^2 - 1 \neq 0$, $4\lambda_1^2 - 1 \neq 0$.

For further investigations it is convenient, as is customary in the discussion of linear anisotropic elasticity, to represent the 21 components of c_{ijkl} by the usual 6×6 symmetric matrix notation [9]

Triclinic symmetry (no plane of symmetry)

$$\left\{ \begin{array}{cccccc} c_{1111} & c_{1122} & c_{1133} & c_{1123} & c_{1113} & c_{1112} \\ & c_{2222} & c_{2233} & c_{2223} & c_{2213} & c_{2212} \\ & & c_{3333} & c_{3323} & c_{3313} & c_{3312} \\ & & & c_{2323} & c_{2313} & c_{2312} \\ & & & & c_{1313} & c_{1312} \\ & & & & & c_{1212} \end{array} \right\} \tag{6.13}$$

Then making use of here derived results we have the following representations:

Monoclinic symmetry (one plane of symmetry)

$$\left\{\begin{array}{cccccc} c_{1111} & c_{1122} & c_{1133} & c_{1123} & 0 & 0 \\ & c_{2222} & c_{2233} & c_{2223} & 0 & 0 \\ & & c_{3333} & c_{3323} & 0 & 0 \\ & & & c_{2323} & 0 & 0 \\ & & & & c_{1313} & c_{1312} \\ & & & & & c_{1212} \end{array}\right\} \tag{6.14}$$

Rhombic or orthotropic symmetry
(Three mutually perpendicular planes of symmetry)
All classes

$$\left\{\begin{array}{cccccc} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & 0 \\ & c_{2222} & c_{2233} & 0 & 0 & 0 \\ & & c_{3333} & 0 & 0 & 0 \\ & & & c_{2323} & 0 & 0 \\ & & & & c_{1313} & 0 \\ & & & & & c_{1212} \end{array}\right\} \tag{6.15}$$

Tetragonal symmetry
Classes 4mm, $\overline{4}$2m, 4/mm

$$\left\{\begin{array}{cccccc} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & 0 \\ & c_{1111} & c_{1133} & 0 & 0 & 0 \\ & & c_{3333} & 0 & 0 & 0 \\ & & & c_{2323} & 0 & 0 \\ & & & & c_{2323} & 0 \\ & & & & & c_{1212} \end{array}\right\} \tag{6.16}$$

Trigonal symmetry
Classes 32, $\overline{3}$m, 3mm

$$\left\{\begin{array}{cccccc} c_{1111} & c_{1122} & c_{1133} & c_{1123} & 0 & 0 \\ & c_{1111} & c_{1133} & -c_{1123} & 0 & 0 \\ & & c_{3333} & 0 & 0 & 0 \\ & & & c_{2323} & 0 & 0 \\ & & & & c_{2323} & c_{1123} \\ & & & & & 1/2(c_{1111} - c_{1122}) \end{array}\right\} \tag{6.17}$$

Transverse isotropy
Hexagonal symmetry
All classes

$$
\left\{
\begin{array}{cccccc}
c_{1111} & c_{1122} & c_{1133} & c_{1123} & 0 & 0 \\
 & c_{1111} & c_{1133} & 0 & 0 & 0 \\
 & & c_{3333} & 0 & 0 & 0 \\
 & & & c_{2323} & 0 & 0 \\
 & & & & c_{2323} & 0 \\
 & & & & & 1/2(c_{1111} - c_{1122})
\end{array}
\right\} \tag{6.18}
$$

Second, we turn now to the case when unit vectors **a** and **m** are in the plane $x_3 = 0$, but neither of them are necessarily in the direction of x_1 axes [1]. Then

$$
\begin{aligned}
& a_i = \delta_{i\alpha} a_\alpha \quad a_\alpha a_\alpha = 1 \quad (a_1 = \cos\Theta,\ a_2 = \sin\Theta) \\
& b_{1i} = \delta_{3i} \\
& b_{2i} = \delta_{2i} \quad b_i = \delta_{i\alpha} e_{\beta\alpha} a_\beta, \quad \alpha, \beta = 1, 2.
\end{aligned} \tag{6.19}
$$

Now the set of conditions of the form (6.6) and (6.7), where n_i and m_i has to be replaced by b_{2i} and a_i, respectively, must be satisfied for any Θ. It then follows that

$$
\begin{aligned}
& c_{1213} = c_{1233} = c_{1333} = c_{2333} = 0 \\
& c_{1111} = c_{2222},\ c_{1313} = c_{2323},\ c_{1133} = c_{2233} \\
& c_{1223} = c_{1322} = -c_{1113},\ c_{1123} = c_{1213} = -c_{2223} \\
& c_{1112} = -c_{2221}
\end{aligned} \tag{6.20}
$$

and

$$
c_{1123} \sin 3\Theta + c_{1113} \cos 3\Theta = 0, \tag{6.21}
$$

$$
\frac{1}{4}(c_{1111} - c_{1122} - 2c_{1212}) \sin 4\Theta - c_{1112} \cos 4\Theta = 0. \tag{6.22}
$$

Further, making use of (6.2-7) (with $\lambda_1 = \cos(\Theta + \alpha)$, $\lambda_2 = \sin(\Theta + \alpha)$, $\alpha \neq 0$) we obtain (6.20) and

$$
c_{1123} \sin 3\Theta + c_{1113} \cos 3(\Theta + \alpha) = 0, \tag{6.23}
$$

$$
\frac{1}{4}(c_{1111} - c_{1122} - 2c_{1212}) \sin 4(\Theta + \alpha) - c_{1112} \cos 4(\Theta + \alpha) = 0. \tag{6.24}
$$

These results are the same as (3.21) and (3.22) in [1]. the determinants of sets of homogeneous equations (6.21) and (6.23), and (6.22) and (6.24) are

$$
\sin 3\alpha \quad \text{and} \quad \sin 4\alpha \tag{6.25}
$$

respectively. Here we have the following cases:

i) $\alpha = \pm\pi/3$ so that $c_{1212} = 1/2(c_{1111} - c_{1122})$, $c_{1112} = 0$

Trigonal symmetry

Classes 3, $\overline{3}$

$$\left\{\begin{array}{cccccc} c_{1111} & c_{1122} & c_{1133} & c_{1123} & c_{1113} & 0 \\ & c_{1111} & c_{1133} & -c_{1123} & -c_{1113} & 0 \\ & & c_{3333} & 0 & 0 & 0 \\ & & & c_{2323} & 0 & -c_{1113} \\ & & & & c_{2323} & c_{1123} \\ & & & & & 1/2(c_{1111}-c_{1122}) \end{array}\right\} \tag{6.26}$$

ii) $\alpha = \pm\pi/4, \pm\pi/2$ so that $c_{1123} = c_{1113} = 0$

Tetragonal symmetry
Classes 4, $\overline{4}$, 4/mm

$$\left\{\begin{array}{cccccc} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & c_{1112} \\ & c_{1111} & c_{1133} & 0 & 0 & -c_{1112} \\ & & c_{3333} & 0 & 0 & 0 \\ & & & c_{2323} & 0 & 0 \\ & & & & c_{2323} & 0 \\ & & & & & c_{1212} \end{array}\right\}. \tag{6.27}$$

These eight distinct elastic symmetry are the only symmetry which we may derive using the method presented here under the assumption that a material possesses, at the beginning of investigation one or two planes of symmetry.

Third. We are going to proceed with our investigation in a case when a material possesses three planes of symmetry. But instead of approaching the problem in pure algebraic way, we used up to now, we may simplify investigation making use of the derived results. In fact, we may assume that, in addition to the symmetric properiety analyzed here, material possesses and additional distinct plane of symmetry defined by its unit normal vector, say **s**, but now in a plane defined by x_1 and x_3 axes. Then all results, derived here, hold if one interchanges the indices 2 and 3 in them. In this way we shall investigate case by case.

Obviousely nothing new will come out in a case of *Rhombic of orthorombic symmetry.* But in a case of Tetragonal symmetry, (6.16), in addition to (1.7), (5.6) and (6.10) one will have (from (6.10)):

$$\begin{aligned} c_{1111} &= c_{3333} \\ c_{2211} &= c_{2233} \\ c_{1212} &= c_{3232} \end{aligned} \tag{6.28}$$

Then a material will possess

Cubic symmetry
All classes

$$\left\{\begin{array}{cccccc} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & 0 \\ & c_{1111} & c_{1122} & 0 & 0 & 0 \\ & & c_{1111} & 0 & 0 & 0 \\ & & & c_{2323} & 0 & 0 \\ & & & & c_{2323} & 0 \\ & & & & & c_{2323} \end{array}\right\} \tag{6.29}$$

The same kind of symmetry one will obtain from *Tetragonal symmetry* (6.27).

Also from *Trigonal symmetries* (6.17) and (2.26), *Transversal - Hexagonal symmetry* (6.18) one will obtain

Isotropic symmetry

$$\left\{\begin{array}{cccccc} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & 0 \\ & c_{1111} & c_{1122} & 0 & 0 & 0 \\ & & c_{1111} & 0 & 0 & 0 \\ & & & \dfrac{c_{1111}-c_{1122}}{2} & 0 & 0 \\ & & & & \dfrac{c_{1111}-c_{1122}}{2} & 0 \\ & & & & & \dfrac{c_{1111}-c_{1122}}{2} \end{array}\right\} \tag{6.30}$$

It is simple matter now to show that material cannot possess more than these ten traditional and distinct elastic material symmetries by planes of symmetry.

7 Conclusion

The method presented here for a determination of planes of symmetry for a linear anisotropic elastic material is an algebraic one. The necessary and sufficient conditions for the existence of symmetry planes are given in several equivalent forms, and are used to determine special coordinate systems where the number of nonzero components in the elasticity tensor is minimized. It is shown that a material cannot possess more than ten traditional and distinct symmetries by planes of symmetry. The approach is, I believe, a very simple one and may be applied, generally speaking, for any tensor of any order. In the application we are interested specifically in those tensors which define physical properties of materials. For instance, we may show by this method that the second-order tensor properties of cubic crystals are isotropic, but its elastic properties, given by fourth-order tensor, are not isotropic. In fact we were dealing only with one part of problem, i.e. we assume the existence of planes of symmetry. The reverse problem, to find them, if they exist, was, extensively investigated in [1]. To my knowledge this is the only algebraical approach which is used.

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O USLOVIMA ZA POSTOJANJE RAVNI SIMETRIJE U SLUČAJU ANIZOTROPNIH ELASTIČNIH MATERIJALA

Rezime: *U radu se razmatra problem određivanja potrebnih i dovoljnih uslova za postojanje ravni simetrije anizotropnog elastičnog materijala. Ovi uslovi su dati u nekoliko ekvivalentnih formi i korišćeni su za određivanje specijalnih koordinatnih sistema gde je broj komponenti različitih od nule u elastičnom tenzoru minimiziran. Na osnovu datog modela pokazano je da elastično telo ima barem šest koordinatnih sistema u odnosu na koje postoji samo 18 elastičnih komponenti različitih od nule i ono ne može da ima više od deset klasičnih i jedinstvenih simetrija po ravnima ravnima simetrije.*

MODIFIKACIJE GEOCENTRIČNE KONSTANTE GRAVITACIJE

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UDK:531.5

Rezime: *Popravljena je formula i brojna vrednost konstante Zemljine gravitacije, poznate pod nazivom Geocentrična konstanta gravitacije. Prethodno je ukazano na nesaglasje o pitanju Univeralne konstante gravitacije: posle modifikacije geocentrične konstante gravitacije popravljene su i konstante glavnih planeta i Meseca.*

Ključne reči: *gravitacione konstante, geocentrična konstanta, sile gravitacije, modifikacija.*

1. FORMULA UNIVERZALNE KONSTANTE GRAVITACIJE

U postnjutnovskom periodu njegova VIII teorema knjige O SISTEMU SVETA o uzajamnom privlačenju dve homogene kugle, i teorema I, II, III o kretnju Jupiterovih satelita, kretanju glavnih planeta oko Sunca, kao i o zadržavanju Meseca na orbiti oko Zemlje, uobličene su pod nazivom *Njutnov zakon gravitacije* formulom

$$F = f\frac{m_1 m_2}{\rho^2} \tag{1}$$

gde je f- univerzalna konstanta gravitacije, a ρ međusobno rastojanje materijalnih tačaka, masa m_1 i m_2.

Brojna vrednost konstante f nije precizno tabulisana. Određivana je eksperimentalno i računski, ali je najčešće upotrebljavaju kao dimenzioni broj $6{,}27\cdot10^{-11}$ kg^{-1} m^3 s^{-2}. Međutim, kako je formula (1) izvedena na osnovu Keplerovih zakona, bilo je moguće odrediti navedenu konstantu formulom

$$f = \frac{4\pi^2 a^3}{(M+m)T^2}\,, \tag{2}$$

gde je M masa Sunca, a m masa planete. Ali po toj formuli je očigledno da je f funkcija mase planete, tj. da je različita za razne planete. To postaje sasvim jasno ako se zapazi da je

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$$\mu = \frac{4\pi^2 a^3}{T^2} \tag{3}$$

konstantna veličina i ne zavisi od masa planeta, s obzirom da količnik

$$\frac{a^3}{T^2} = k = const. \tag{4}$$

prema trećem Keplerovom zakonu, ima jednu te istu vrednost za sve planete Sunčevog sistema. Prema tome, formula (2) može se napisati u obliku

$$F = \frac{\mu}{M+m}\frac{Mm}{\rho^2}. \tag{5}$$

Suočavajući se sa teškoćama, prišlo se zanemarivanju masa planeta u odnosu na daleko veću masu Sunca, pa je formula (2) svedena na

$$f^* = \frac{\mu}{M} = \frac{4\pi^2 a^3}{MT^2}\ , \tag{6}$$

gde se sada podrazumeva da je (iako to nije) masa Sunca M u koeficijentu proporcionalnosti f konstanta, pa je i f^* konstanta. Međutim, sledeći račun (Vidi, na primer [3]), pokazuje da ni to ne dovodi do strogog zaključka da univerzalna gravitaciona konstanta ima jednu te istu brojnu vrednost. Ne mali broj autora, visokih učenih znanja i zvanja i uglednih imena, upoređuju Njutnovu silu gravitacije (1) sa silom Zemljine teše mg, te nalaze međusobnu zavisnost između ubrzanja g i konstante gravitacije f. To će reći, da formulu (1) proširuju ne samo na planetarni sistem, nego i na bilo koja dva tela. Evo kako to izgleda prema [1]. Sila $F = f^* \frac{M_z m}{R^2}$ kojom Zemlja mase M_z privlači neku materijalnu tačku mase m na površini Zemlje, jednaka je težini mg, pri čemu je g=9,78 ms^{-2} do 9,83 ms^{-2}. Kako je, prema tome,

$$f^* \frac{M_z m}{R^2} = mg\ , \tag{7}$$

sledi da je gravitacionu konstantu f^* moguće tačno odrediti onoliko, koliko su tačne vrednosti za poluprečnik i masu Zemlje; neka je to $6{,}67 \cdot 10^{-11}$ kg^{-1} m^3 s^{-2}. Međutim, u stručnoj literaturi (vidi na primer [2]) o kretanju veštačkih satelita, koriste se znanto drugačije vrednosti konstanata gravitacije. Zaista, ako striktno primenimo formule (6) i (5)

$$F = \frac{\mu}{M_z} \frac{M_z m}{R^2} = \frac{\mu m}{R^2}\ mg\ , \qquad (8)$$

dobija se

$$\mu = 3{,}9860\ m^3 s^{-2} x 10^{14}$$

a to je znatno, skoro dva puta, manje od brojne vrednosti široko usvojene univerzalne gravitacione konstanate. Da bi se otklonila ta nesaglasnost, u nebeskoj mehanici se uvode takozvane *karakteristične konstante gravitacije,* [1] koje se definišu sledećim formulama:

$$\lambda_i = f^* m_i . \qquad (9)$$

Da bi se to dovelo u razuman algebarski sklad sa formulama (3) ili (4), to bi značilo: uzme li se za masu m_1 masa M tela u odnosu na koje se posmatra kretanje materijalne tačke mase $m_2=m,$ formula sile privlačenja (1) se zapisuje u sledećem obliku

$$F = \frac{\mu m}{(1+\frac{m}{M_z})\rho^2} = \lambda^* \frac{m}{\rho^2}. \qquad (10)$$

Ako se m_1 uzme za masu Zemlje M_z i sila gravitacije F izjednači sa silom teže dobija se

$$\lambda^* \frac{m}{R^2} = mg \rightarrow \lambda^* = gR^2. \qquad (11)$$

To znatno uproštava određivanje konstante, ali samo približnom tačnošću za izabrane brojne vrednosti g i R, jer se zna da i jedna i druga vleičina zavise od geometrijskih i kinematičkih parametara. Da bi formulu sile (10), kojom Zemlja privlači satelit, kao materijalnu tačku, doveli u sklad sa izvedenom formulom (2) i izveli formulu geocentrične konstante gravitacije postupimo u daljem kao što to sledi.

2. GEOCENTRIČNA SILA GRAVITACIJE

Pod ovim podnaslovom podrazumevamo silu (1) uzajamnog dejstva bilo koje dve materijalne tačke, u kojoj je $m_1 = M_z$ masa Zemlje, a m masa satelita, kao materijalna tačka koja se kreće oko Zemlje po Keplerovim zakonima. U tom Zemljinom slučaju formula (1) se konkretizuje kao

$$F=\frac{4\pi^2a^3}{(M_z+m)T^2}\,\frac{M_z m}{\rho^2}=\mu\left(\frac{M_z}{M_z+m}\right)\frac{m}{\rho^2}=GE\,\frac{m}{\rho^2}\;;\;GE\overset{def}{=}\mu\left(\frac{1}{1+\frac{m}{M_z}}\right). \tag{12}$$

Dalje se može pisati:

$$\lambda\equiv GE=\mu\frac{1}{1+\frac{m}{M_z}}=\mu\varepsilon \tag{13}$$

gde je

$$\varepsilon=\frac{M_z}{M_z+m}=\frac{1}{1+\frac{m}{M_z}}. \tag{14}$$

Ako, na primer, materijalna tačka ima masu od jednog kilograma onda je

$$GE = \mu\cdot 0{,}999999$$

jer je

$1+\frac{1}{M_z}=1{,}0000000000000000000000000001$. Dakle, po želji visokom tačnošću moguće je odrediti faktor proporcionalnosti *GE*. Ali bez obzira na malenkost količnika ne može se konstatovati da konstanta gravitacije Zemlje *GE* ima jednu te istu brojnu vrednost. To se lako zaključuje ako se umesto mase $m=1$ uzme u obzir masa najvećeg satelita – Meseca, čija je masa $m_M = 0{,}0123\ M_z$; srednje rastojanje a=3,84·10^8 m,

$$\varepsilon = 0.987849,\quad \mu = 4{,}013534\cdot 10^{14}, \tag{15}$$

pa je geocentrična konstanta gravitacije za Mesec

$$GE = 3{,}96476590\cdot 10^{14}\ \mathrm{m^3s^{-2}}\,.$$

Još jasnije se pokazuje da se geocentrična konstanta gravitacije razlikuje, istina veoma malo, od objekta do objekta, kao i od standardne njene usvojene brojne vrednosti $GE = GF$= 3,986005·10^{14} m^{3}s^{-2} , u primeni na veštačke satelite Zemlje.

Primenjeno na 12 veštačkih satelita Kosmos-N, čije su brojne vrednosti apogrja, perigrja i vreme obilaženja Zemlje dati u knjizi “Astronomija i kosmonautika” od S.I. Selešnikova (1967), za geocentričunu konstantu gravitacije dobijaju se sledeći podaci:

Veštački satelit	Godina izbacanja	Geocentrična gravitaciona konstanta 10^{14} m s
Kosmos-1	1962	4.008933
Kosmos-11	1962	4.005365
Kosmos-21	1963	4.008515
Kosmos-31	1964	4.012660
Kosmos-41	1964	3.993121
Kosmos-51	1964	4.007137
Kosmos-61	1965	4.008320
Kosmos-71	1965	3.997788
Kosmos-81	1965	3.943948
Kosmos-91	1965	4.008447
Kosmos-101	1966	4.014837
Kosmos-127	1966	4.399788

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MODIFICATIONS OF GEOCENTRIC GRAVITY CONSTANT

Summary: *Expression and numerical value of Earth gravity constant, otherwise known as Geocentric gravity constant, has been corrected. Prior to that, conflicting issues regarding Universal gravity constant have been emphasized; after modifications of Geocentric gravity constant corrections have been made to constants of major planets and the Moon.*

Key words: *gravity constants, Geocentric constant, Gravity force, modification*

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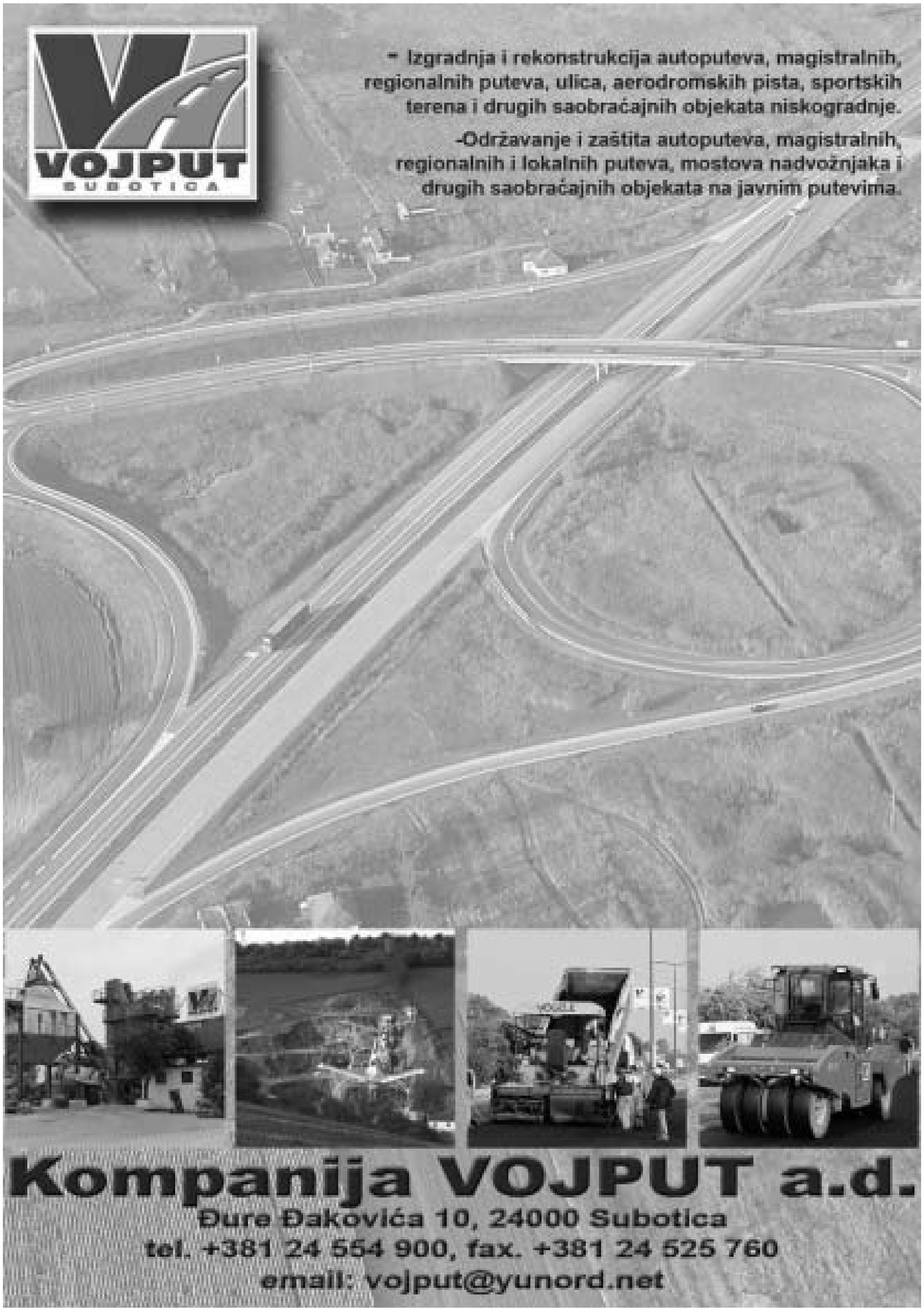
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